

Players' Patience and Equilibrium Payoffs in the Baron–Ferejohn Model

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Abstract

This paper investigates a generalized Baron–Ferejohn model with different discount factors, different recognition probabilities and q -majority rule. In the paper, it is shown that if players are sufficiently patient, recognition probabilities are similar and the voting rule is not unanimous, each player's equilibrium payoff is inversely proportional to the ratio of the player's discount factor to the harmonic mean of all players' discount factors. This result implies the followings: (i) A less patient player obtains a greater payoff; (ii) As a player slightly becomes more patient, her payoff becomes smaller; (iii) The equilibrium payoffs do not depend on recognition probabilities; and (iv) They do not also depend on q .

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1. Introduction

Baron and Ferejohn (1989) introduced a sequential bargaining model in which a randomly-selected proposer offers a proposal, which is put to a vote, and if a majority of players accept it, the game terminates, and otherwise, the procedure is repeated. Banks and Duggan (2000) generalized the Baron and Ferejohn model (hereafter B-F model) and showed that there exists an equilibrium in the model. Eraslan (2002) proved that equilibrium payoffs are uniquely determined in a generalized B-F model, which is a special case of Banks and Duggan's model.

This paper shows that in a generalized B-F model with different discount factors, different recognition probabilities and q -majority rule, which is the same as Eraslan's model, if players are sufficiently patient, recognition probabilities are similar and the voting rule is not unanimous, a player's equilibrium payoff is equal to $\frac{H(\delta)}{\delta_i} \frac{1}{n}$, where δ_i is the player's discount factor, $H(\delta)$ is the harmonic mean of players' discount factors, and n is the number of players. This result implies the followings: (i) A less patient player obtains a greater payoff; (ii) As a player slightly becomes more patient, her payoff becomes smaller; (iii) The equilibrium payoffs do not depend on recognition probabilities; and (iv) They do not also depend on q .

(i) and (ii) contrast with the result of standard bargaining models, in which more patient players obtain greater payoffs. Under q -majority rule, less patient responders, whose approval seems cheaper, can belong to winning coalitions with higher probabilities, which is the driving force of (i) and (ii). On the other hand, it is also shown that under the unanimity rule, more patient players obtain greater payoffs since every responder belongs to winning coalitions with certainty.

The intuition of (iii) is as follows: Under q -majority rule, a player with high recognition probability (a) belongs to winning coalitions with low probability when she is a responder since her approval seems expensive but (b) enjoys the agenda-setting power with high probability. (a) and (b) offset each other, and thus recognition probabilities do not affect equilibrium payoffs. On the other hand, under the unanimity rule, the equilibrium payoffs are monotonic with recognition probabilities.

Under larger q , a proposer's pie is smaller, but a responder belongs to winning coalitions with higher frequency. (iv) implies that these two effects exactly offset each other.

(i) and (ii) are related to Kawamori (2004), Yildirim (2005), Eraslan (2002) and Harrington (1990). Kawamori (2004) analyzed a three-player B-F model with different discount factors and analytically computed equilibrium payoffs for each discount factor profile in $(0, 1)^3$. The computation implies that when all players are as patient as each other, equilibrium payoffs are decreasing with respect to discount factors, which implies (i) and (ii) in the three-player model. Yildirim (2005), which generalized the B-F model by endogenizing recognition probabilities, presented a three-player example in which equilibrium payoffs are not increasing with respect to discount factors in the B-F model with different discount factors (Case 1 in Example 1). Eraslan (2002) showed that *discounted* equilibrium payoffs¹ are monotonic with discount factors.² Harrington (1990) showed that in a generalized B-F model with different risk preferences, if players' risk preferences are similar and a voting rule is not the unanimity rule, a more risk-averse player's probability distribution over her pie induced by an equilibrium first-order stochastically dominates a less risk-averse player's. (i) and (ii) may also be related to Haan

¹A player's discounted equilibrium payoff is her discount factor times her equilibrium payoff.

²Eraslan stated that equilibrium payoffs (not *discounted* equilibrium payoffs) are monotonic with discount factors, which is, however, false as Kawamori (2004), Yildirim (2005) and this paper imply.

and Kooreman (2003) and Piccione and Rubinstein (2004), which showed that a seemingly beneficial property does not necessarily lead to a good result.

The paper is organized as follows: Section 2 describes a generalized B-F model, and Section 3 presents results.

2. The model

Consider the following noncooperative bargaining game, which is a generalized B-F model.

Let $N \equiv \{1, \dots, n\}$ for some $n \in \mathbb{N}$ such that $n \geq 3$. Define Δ and P as $\Delta \equiv (0, 1)^n$ and $P \equiv \{(p_k)_{k \in N} \in \mathbb{R}_+^n \mid \sum_{k \in N} p_k = 1\}$, respectively. For $\delta \equiv (\delta_k)_{k \in N} \in \Delta$, $p \equiv (p_k)_{k \in N} \in P$ and $q \in \{2, \dots, n\}$, let $G(\delta, p, q)$ denote an extensive form game defined as follows:

The set of players is N . Players sequentially make a split-the-pie bargain under q -majority rule. Let $X \equiv \{(x_k)_{k \in N} \in \mathbb{R}_+^n \mid \sum_{k \in N} x_k = 1\}$. X is the set of distributions of the one-unit divisible pie. In each stage game, bargaining proceeds as follows: (i) Nature selects a player $i \in N$ as a proposer with probability p_i ; (ii) The selected proposer i offers a proposal $x \in X$; (iii) Every player j , sequentially according to some predetermined order, votes on the proposal x , i.e., announces either accepting or rejecting it. Then, if more than q players accept the proposal, the proposal is implemented and the game ends. Otherwise, the procedure is repeated from (i). Each player's payoff is the pie distributed to herself and each player discounts the future pie. That is, player i 's payoff is equal to $\delta_i^{t-1} x_i$ when $x \equiv (x_k)_{k \in N}$ is implemented at the t -th stage.

In this paper, we use behavior strategies. The equilibrium concept employed in the paper is the stationary subgame perfect equilibrium (SSPE), which is the subgame perfect equilibrium such that each player takes the same actions in every stage. In this game, there exists an equilibrium³ and equilibrium payoffs are uniquely determined.⁴ Thus, we can denote player i 's equilibrium payoff of $G(\delta, p, q)$ by $v_i^\circ(\delta, p, q)$ for $i \in N$, $\delta \in \Delta$, $p \in P$ and $q \in \{2, \dots, n\}$.

Finally, introduce the following notations: For any family $a \equiv (a_k)_{k \in K}$, let $\text{pr}_k a \equiv a_k$. For any vector $a \equiv (a_k)_{k \in K}$, let $H(a) \equiv \left(\frac{1}{\text{card } K} \sum_{k \in K} a_k^{-1}\right)^{-1}$, which is the harmonic mean of a . For $m \in \mathbb{N}$, let $\mathbf{1}_m$ ($\mathbf{0}_m$) be an m -tuple such that $\text{pr}_k \mathbf{1}_m = 1$ ($\text{pr}_k \mathbf{0}_m = 0$) for all $k \in \{1, \dots, m\}$. For $m \in \mathbb{N}$, $\epsilon > 0$ and $a \in \mathbb{R}^m$, let $B_\epsilon^m(a)$ be an ϵ -open ball of a on \mathbb{R}^m .

3. Results

The following theorem is the main result in this paper. The theorem means that under q -majority rule ($q < n$), if every player is sufficiently patient and all players' recognition probabilities are similar, each player's equilibrium payoff is inversely proportional to the ratio of the player's discount factor to the harmonic mean of all players' discount factors.

Theorem 1. *Take any $q \in \{2, \dots, n-1\}$. There exists $\epsilon, \epsilon' > 0$ such that for all $\delta \equiv (\delta_k)_{k \in N} \in B_\epsilon^n(\mathbf{1}_n) \cap \Delta$ and all $p \equiv (p_k)_{k \in N} \in B_{\epsilon'}^n(\frac{1}{n}\mathbf{1}_n) \cap P$, for all $i \in N$, $v_i^\circ(\delta, p, q) = \frac{H(\delta)}{\delta_i} \frac{1}{n}$.*

Remark. Even if “ $\delta \in B_\epsilon^n(\mathbf{1}_n) \cap \Delta$ ” is replaced with “ $\delta \in B_\epsilon^n(\delta^*\mathbf{1}_n) \cap \Delta$ ” for some $\delta^* \in [0, 1)$, the theorem holds.

³See Banks and Duggan (2000).

⁴See Eraslan (2002).

Proof. Let $S \equiv \mathbb{R}^n \times \left(0, \frac{1}{q}\right) \times (0, 1)^{n-1} \times \mathbb{R}^{2n}$. Define C^1 function $f : S \rightarrow \mathbb{R}^{2n}$ as follows: For $z \equiv (v, u, r_{-1}, \delta, p) \equiv \left((v_k)_{k \in N}, u, (r_k)_{k \in N \setminus \{1\}}, (\delta_k)_{k \in N}, (p_k)_{k \in N}\right)$, for $i \in N$,

$$\begin{aligned} \text{pr}_i f(z) &= v_i - p_i \{1 - (q-1)u\} - \left\{ p_{i+1}r_{i+1} + p_{i+2}(1 - r_{i+2}) + \sum_{k=3}^q p_{i+k} \right\} u \\ \text{pr}_{i+n} f(z) &= \delta_i v_i - u, \end{aligned}$$

where $r_1 \equiv \frac{1}{2}$.⁵ Let $z^* \equiv (v^*, u^*, r_{-1}^*, \delta^*, p^*) \equiv \left(\frac{1}{n}\mathbf{1}_n, \frac{1}{n}, \frac{1}{2}\mathbf{1}_{n-1}, \mathbf{1}_n, \frac{1}{n}\mathbf{1}_n\right)$. Obviously, $f(z^*) = \mathbf{0}_{2n}$ holds and the derivative of $f(z)$ with respect to (v, u, r_{-1}) evaluated at $z = z^*$ is a full rank matrix. Thus, the Implicit Function Theorem implies that there exists $\bar{\epsilon} > 0$ and C^1 function $g : B_{\bar{\epsilon}}^{2n}(\delta^*, p^*) \rightarrow \mathbb{R}^{2n}$ such that $(g(\delta, p), \delta, p) \in S$ for all $(\delta, p) \in B_{\bar{\epsilon}}^{2n}(\delta^*, p^*)$, $(v^*, u^*, r_{-1}^*) = g(\delta^*, p^*)$ and $f(g(\delta, p), \delta, p) = \mathbf{0}_{2n}$ for all $(\delta, p) \in B_{\bar{\epsilon}}^{2n}(\delta^*, p^*)$. Let $\hat{v}_i(\delta, p) \equiv \text{pr}_i g(\delta, p)$ for $i \in N$, $\hat{u}(\delta, p) \equiv \text{pr}_{n+1} g(\delta, p)$, $\hat{r}_i(\delta, p) \equiv \text{pr}_{i+n} g(\delta, p)$ for $i \in N \setminus \{1\}$ and $\hat{r}_1(\delta, p) \equiv \frac{1}{2}$, for $(\delta, p) \in B_{\bar{\epsilon}}^{2n}(\delta^*, p^*)$. Note that $\hat{r}_1(\delta, p) = r_1$.

Consider ϵ and ϵ' such that $B_{\epsilon}^n(\delta^*) \times B_{\epsilon'}^n(p^*) \subset B_{\bar{\epsilon}}^{2n}(\delta^*, p^*)$. Take any $\delta \in B_{\epsilon}^n(\delta^*) \cap \Delta$ and $p \in B_{\epsilon'}^n(p^*) \cap P$. Consider strategy profile σ defined as follows:

- Every player i proposes $(x_k)_{k \in N}$ such that

$$x_k \equiv \begin{cases} 1 - (q-1)\hat{u}(\delta, p) & \text{if } k = i \\ \hat{u}(\delta, p) & \text{if } k \in \{i-1, i-3, \dots, i-q\} \\ 0 & \text{otherwise} \end{cases}$$

with probability $\hat{r}_i(\delta, p)$ and $(x'_k)_{k \in N}$ such that

$$x'_k \equiv \begin{cases} 1 - (q-1)\hat{u}(\delta, p) & \text{if } k = i \\ \hat{u}(\delta, p) & \text{if } k \in \{i-2, i-3, \dots, i-q\} \\ 0 & \text{otherwise} \end{cases}$$

with probability $1 - \hat{r}_i(\delta, p)$.

- Every player i accepts a proposal $(y_k)_{k \in N}$ with probability 1 if $y_i \geq \hat{u}(\delta, p)$ and rejects it otherwise.

Let V_i be player i 's payoff by σ . Then,

$$V_i = p_i \{1 - (q-1)\hat{u}(\delta, p)\} + \left\{ p_{i+1}\hat{r}_{i+1}(\delta, p) + p_{i+2}(1 - \hat{r}_{i+2}(\delta, p)) + \sum_{k=3}^q p_{i+k} \right\} \hat{u}(\delta, p).$$

For $i \in N$, $V_i = \hat{v}_i(\delta, p)$ since $\text{pr}_i f(g(\delta, p), \delta, p) = 0$. Notice that $\text{pr}_{i+n} f(g(\delta, p), \delta, p) = 0$ for all $i \in N$. Then, $\delta_i V_i = \delta_i \hat{v}_i(\delta, p) = \hat{u}(\delta, p)$. Consider the unimprovability of player i 's strategy of σ . Obviously, player i 's voting action of σ is unimprovable. Consider player i 's proposing action. Player i 's proposal of σ is obviously optimal among proposals to pass given voting actions of σ .

⁵For $i, j \in \mathbb{Z}$, on indices representing players, we regard i as identical with j if $i \equiv j \pmod{n}$.

Player i , at her proposing nodes, obtains a payoff of $1 - (q - 1) \hat{u}(\delta, p)$ by the proposal of σ and a payoff of $\delta_i V_i = \hat{u}(\delta, p)$ by proposals not to pass given voting actions of σ . The former is greater than the latter since $\hat{u}(\delta, p) < \frac{1}{q}$. Thus, player i 's proposing action of σ is unimprovable. Since i is arbitrary, σ is unimprovable. Hence, the One Deviation Principle implies that σ is an SPE. σ is obviously stationary. Therefore, σ is an SSPE. Notice that $\sum_{k \in N} V_k = 1$ since there is no delay in equilibrium. Then, $\sum_{k \in N} \frac{1}{\delta_k} \hat{u}(\delta, p) = 1$. Thus, $\hat{u}(\delta, p) = H(\delta) \frac{1}{n}$. $V_i = \frac{H(\delta)}{\delta_i} \frac{1}{n}$. Since σ is an SSPE, $v_i^\circ(\delta, p, q) = V_i = \frac{H(\delta)}{\delta_i} \frac{1}{n}$. **Q.E.D.**

The theorem implies the following four corollaries under similar discount factors and recognition probabilities.

The first corollary says that a less patient player's equilibrium payoff is greater than a more patient player's.

Corollary 1. *Take any $q \in \{2, \dots, n - 1\}$. There exists $\epsilon, \epsilon' > 0$ such that for all $\delta \equiv (\delta_k)_{k \in N} \in B_\epsilon^n(\mathbf{1}_n) \cap \Delta$ and all $p \in B_{\epsilon'}^n(\frac{1}{n}\mathbf{1}_n) \cap P$, for all $i, j \in N$, $v_i^\circ(\delta, p, q) \gtrless v_j^\circ(\delta, p, q)$ if and only if $\delta_i \gtrless \delta_j$.*

The second corollary says that as a player becomes slightly more patient, the player's equilibrium payoff decreases.

Corollary 2. *Take any $q \in \{2, \dots, n - 1\}$. There exists $\epsilon, \epsilon' > 0$ such that for all $\delta \equiv (\delta_k)_{k \in N} \in B_\epsilon^n(\mathbf{1}_n) \cap \Delta$ and all $p \in B_{\epsilon'}^n(\frac{1}{n}\mathbf{1}_n) \cap P$, for all $i \in N$, $\frac{\partial v_i^\circ(\delta, p, q)}{\partial \delta_i} < 0$.*

In standard bargaining games, a more patient player has a stronger bargaining power and thus obtains a larger payoff, which contrasts with Corollaries 1 and 2. In the Baron-Ferejohn model with q -majority rule ($q < n$), a proposer can make a proposal pass by distributing only $q - 1$ responders their continuation payoffs respectively and winning their approval. Thus, the proposer wants to form winning coalitions with responders who seem to obtain small continuation values. Therefore, less patient responders can belong to winning coalitions with higher probabilities, which is the driving force of Corollaries 1 and 2.

Example 1. Consider the case that $n = 3$. Let $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $q = 2$. Take any $\delta \equiv (\delta_k)_{k \in N} \in (0, 1)^3$ such that $\delta_i \geq \frac{2}{3}$ for all $i \in N$. Then, consider game $G(\delta, p, q)$. In this game, the following strategy profile is an SSPE:

- Every player i proposes $(x_k)_{k \in N}$ such that $x_i \equiv 1 - \frac{H(\delta)}{3}$, $x_{i+1} \equiv \frac{H(\delta)}{3}$ and $x_{i+2} = 0$ with probability $r_i \equiv \frac{1}{\delta_{i+1}} - \frac{1}{\delta_{i+2}} + \frac{1}{2} \in (0, 1)$, and $(x'_k)_{k \in N}$ such that $x'_i = 1 - \frac{H(\delta)}{3}$, $x'_{i+1} = 0$ and $x'_{i+2} \equiv \frac{H(\delta)}{3}$ with probability $1 - r_i$.
- Every player i accepts a proposal $(y_k)_{k \in N}$ with probability 1 if $y_i \geq \frac{H(\delta)}{3}$ and rejects it with probability 1 otherwise.

Note that r_i is the probability that player $i + 1$ belongs to winning coalitions when player i is a proposer and the probability is decreasing in player $i + 1$'s discount factor. Player i 's equilibrium payoff V_i is computed as $V_i = \frac{1}{3} \left(1 - \frac{H(\delta)}{3} \right) + \frac{1}{3} (1 - r_{i+1}) \frac{H(\delta)}{3} + \frac{1}{3} r_{i+2} \frac{H(\delta)}{3} = \frac{H(\delta)}{\delta_i} \frac{1}{3}$.

On the other hand, if $q = n$, all responders' approval is necessary for a proposal to pass. Thus, every responder, however patient she is, can be distributed her continuation value. Therefore, less patient players have no advantage and obtain smaller payoffs. Indeed, for any $\delta \equiv (\delta_k)_{k \in N} \in \Delta$ and any $p \equiv (p_k)_{k \in N} \in P$, player i 's equilibrium payoff of $G(\delta, p, n)$ is calculated as $v_i^\circ(\delta, p, n) = \frac{H((1-\delta_k)/p_k)_{k \in N}}{(1-\delta_i)/p_i} \frac{1}{n}$,⁶ which implies that under the same recognition probability, $v_i^\circ(\delta, p, n) \gtrless v_j^\circ(\delta, p, n)$ if and only if $\delta_i \gtrless \delta_j$.

The third corollary says that the equilibrium payoffs are determined independent of recognition probabilities.

Corollary 3. *Take any $q \in \{2, \dots, n-1\}$. There exists $\epsilon, \epsilon' > 0$ such that for all $\delta \in B_\epsilon^n(\mathbf{1}_n) \cap \Delta$ and all $p, p' \in B_{\epsilon'}^n(\frac{1}{n}\mathbf{1}_n) \cap P$, for all $i \in N$, $v_i^\circ(\delta, p, q) = v_i^\circ(\delta, p', q)$.*

Each player's recognition probability has two effects on her equilibrium payoff, between which there is a tradeoff, under q -majority rule ($q < n$). One is that a player with high recognition probability belongs to winning coalitions with low probability when she is a responder since her approval seems expensive. The other is that a player with high recognition probability enjoys the agenda-setting power with high probability. These two effects offset each other and thus recognition probabilities do not affect equilibrium payoffs.⁷ On the other hand, as calculated above, under the unanimity rule, the equilibrium payoffs are monotonic with recognition probabilities. This is because the first effect vanishes under the unanimity rule.

The fourth corollary says that the equilibrium payoffs are determined independent of voting rules.

Corollary 4. *Take any $q, q' \in \{2, \dots, n-1\}$. There exists $\epsilon, \epsilon' > 0$ such that for all $\delta \in B_\epsilon^n(\mathbf{1}_n) \cap \Delta$ and all $p \in B_{\epsilon'}^n(\frac{1}{n}\mathbf{1}_n) \cap P$, for all $i \in N$, $v_i^\circ(\delta, p, q) = v_i^\circ(\delta, p, q')$.*

Under larger q , a proposer's pie is smaller, but a responder belongs to winning coalitions with higher frequency. The corollary implies that these two effects exactly offset each other.

Finally, we remark on some possible extensions: (i) Since players' discount factors significantly affect equilibrium payoffs, each player may want to decide her own patience endogenously. Therefore, it is necessary to endogenize discount factors. (ii) It is natural that a player does not know how patient the other players are. Therefore, treating a player's discount factor as her private information is a more realistic approach.

⁶ $v_i^\circ(\delta, p, n)$ is also written as $v_i^\circ(\delta, p, n) = \frac{H_p((1-\delta_k)_{k \in N})}{1-\delta_i} p_i$, where $H_p((1-\delta_k)_{k \in N})$ is the harmonic mean of $(1-\delta_k)_{k \in N}$ weighted by p .

⁷The two effects *completely* offset each other. The reason is as follows: Consider the case that every player has the same discount factor. Let each player's recognition probability be $\frac{1}{n}$. Then, let player 1's recognition probability marginally increase and the other players' uniformly decrease. Suppose that player 1's equilibrium payoff increases by the change of recognition probabilities. Then, player 1 belongs to winning coalitions with probability 0 when she is not a proposer. Thus, the former effect discontinuously decreases her payoff. On the other hand, the latter effect just marginally increases her payoff. Hence, player 1's payoff must decrease, which is a contradiction. Suppose that player 1's equilibrium payoff decreases by the change of recognition probabilities. Then, player 1 belongs to winning coalitions with probability 1 when she is not a proposer. Thus, the former effect increases her payoff. On the other hand, the latter effect also increases her payoff. Hence, player 1's payoff must increase, which is a contradiction. Therefore, the change of recognition probabilities must not affect player 1's equilibrium payoff.

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