A new proxy of the average volatility of a basket of returns: A Monte Carlo study

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Abstract

The volatility of returns plays a pivotal role in modern finance and an accurate evaluation of this parameter is crucial in portfolio and risk management decisions. Until quite recent it was common practice in the literature to use the squared return as proxy of volatility. However, as pointed out by several authors, this measure of volatility includes a large noisy component. In this paper we propose a procedure, based on a generalized dynamic factors model methodology, to obtain a more accurate estimate of volatility of a basket of returns.

The views expressed in the note are those of the authors and do not involve the responsibility of the Bank of Italy.

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1 Introduction

The volatility of returns plays a pivotal role in modern finance and an accurate evaluation of this parameter is crucial in portfolio and risk management decisions.

It is well known that the volatility is a latent variable. Until quite recent it was common practice in the literature to use the squared return as proxy of volatility (see, for example, Pagan and Schwert (1990) and West and Cho (1995)). However, as pointed out by Anderesen and Bollerslev(1998), this measure of volatility includes a large noisy component (the squared returns are contaminated with highly non-Gaussian measurement error). These authors proposed a different proxy, the "realized volatility", based on the intraday returns.[1] Since its introduction, the realized volatility has been considered the best estimator for the latent variable. However, intraday data sets might be unavailable or not readily available or too short and quotes might be missing or corrupt. In this paper we propose an alternative proxy of the daily average volatility of a basket of returns, representative of a market or a segment of market, that does not require the intraday returns and could be utilised to address the situations where high-frequency price data are unavailable.

This proxy is derived by using a procedure based on generalised dynamic factors models methodology, proposed by Forni et al. (2000, 2001 and 2004). Under the dynamic factor model approach, we suppose that volatility of each returns series is decomposable into two mutually orthogonal components, a common component, driven by a small number of latent shocks or factors, and an idiosyncratic component. Thus, the proxy of volatility is obtained by aggregating the common components.

We consider n assets in a basket. The spot price of each asset i, on day t, is denoted by P_{it}, and its return is defined as

$$r_{it} = \log P_{it} - \log P_{it-1}$$

. We shall assume throughout that the conditional second moments of the r_{it} process exist. The volatility, denoted with σ^2_{it}, is defined as the conditional variance of the return r_{it}, that is

$$\sigma^2_{it} \equiv var(r_{it}|I_{t-1})$$

where I_{t-1} is the σ-algebra induced by variables that are observed at time $t-1$.

[1] This approach has been popularized recently in a collection of papers such Andersen et al. (2000, 2001)

For the basket as a whole, the volatility is measured as the average volatility across all returns, that is

$$\sigma_t^2 = \sum w_{i,t}\sigma_{it}^2$$

where the weights $w_{i,t}$ are proportional to market value of asset i on day t. It is important to note that this measure does not coincide with the volatility of the market index, defined as

$$I_{Mt} = \sum w_{i,t} r_{it}$$

It is clear that the parametric specification for σ_{it}^2 depends on the model used to describe the process generator of returns.

The search for model specification is always guided by stylized facts. Some typical features of returns series are now well documented. The tails of the distributions of these series are fatter than the tails of the normal distribution, the volatility is time-varying and is highly persistent[2], and squared returns exhibit pronounced serial correlation whereas little or no serial dependence can be detected in the return process itself. A class of models reproducing adequately such stylized facts is that of stochastic volatility (hereafter SV) models introduced by Taylor (1986).[3] These models represent a natural alternative to the GARCH family of time-varying volatility models (Engle (1982) and Bollerslev (1986)). We remember that one of the problems in GARCH-type models is that the variance equation does not contain an innovation. The volatility changes deterministically. The SV models, instead, assume the variance to be a random variable. The assumption that the volatility changes stochastically rather than deterministically has a intuitive appeal. Thus, in this paper, we suppose that any returns series follows an appropriate SV model.

The remainder of the article is organized as follows. Section 2 introduces the SV models utilized in the paper. In Section 3 we present a new proxy of volatility of a return index. Section 4 contains a simulation-based comparison between the proposed procedure to estimate the volatility and the proxy based on squared returns.

[2]The persistence of volatility implies that there will be volatility clustering: high volatility periods tend to be followed by high volatility periods, and similarly for low volatility days

[3]For a discussion of theoretical details about SV models see Shephard (1996) and Ghysels et al. (1996)

2 The stochastic volatility models of returns

In this section we describe three different SV models considered in the study, namely the basic SV model (M1), the SV-t model (M2) and the SV-t model with leverage (M3).

2.1 The basic model

A standard formulation of the SV model for daily returns is given by

$$\begin{aligned} r_{it} &= \sigma_{it} z_{it} \\ z_{it} &\sim n.i.i.d.(0,1) \\ \sigma_{it} &= exp(v_{it}/2) \\ v_{it} &= \phi_i v_{it-1} + \sigma_{i\epsilon}\epsilon_{it} \\ \epsilon_{it} &\sim n.i.i.d.(0,1) \end{aligned}$$

The errors processes ϵ_{it} and z_{it} are mutually and serially independent with mean zero and unit variance. The persistence parameter ϕ_i of the autoregressive process is restricted to be positive and smaller than one to ensure stationarity. The parameter $\sigma_{i\epsilon}$ measures the standard deviation of volatility shocks (the volatility of the log-volatility).

For $|\phi_i| < 1$ the unconditional distribution of v_{it} is normal with mean zero and variance $\sigma_{i\epsilon}^2/(1-\phi_i^2)$. The mean and variance of returns are respectively

$$\begin{aligned} E(r_{it}) &= 0 \\ var(r_{it}) &= exp\left(\frac{1}{2}\frac{\sigma_{i\epsilon}^2}{1-\phi_i^2}\right) \end{aligned}$$

The kurtosis is

$$kurtosis(r_{it}) = 3\ \frac{E(r_{it}^4)}{E(r_{it}^2)^2} = 3\ exp\left(\frac{\sigma_{i\epsilon}^2}{1-\phi_i^2}\right)$$

which demonstrates that the SV model implies excess kurtosis in the series r_{it}. This is consistent with the observed leptokurtosis of the empirical distribution of returns series.

For the sequel it is important to note that $\log z_{it}^2$ follows the $\log(\chi_{(1)}^2)$ distribution and that

$$E(\log z_{it}^2) = -1.27$$

and

$$var(\log z_{it}^2) = \frac{\pi^2}{2} = 4.93$$

See Wishart (1947) and Abramovitz and Stegun (1970).

The appeal of the basic stochastic volatility model is its simplicity and ease of interpretation. However, a number of studies has found that the SV model has to be far more complicated if it has to actually fit the data (see, for example, Kim, Shephard, and Chib (1998), Fridman and Harris (1998) and Liesenfeld and Jung (2000)). We extend the basic model to allow for fat-tails in the conditional distribution of the returns, and for so-called "leverage effect".

2.2 Stochastic volatility model with heavy-tailed distribution

Although the normality assumption for z_{it} coupled with time-varying volatility implies that unconditional distribution of r_{it} has fatter tails than the normal, this is typically not sufficient to account for all of the mass in the tails in the distributions of returns. In many situations the distribution of z_{it} is far from being normal.[4] In this cases a popular choice for z_{it} is the normalized t-distribution with $p > 3$ degrees of freedom, with the normalization such that the variance of z_{it} is unity. Thus, the model with fatter tails than the normal (cfr. Taylor (2005, p. 291)), called the SV-t model, is given by

$$\begin{aligned}
r_{it} &= \sigma_{it} z_{it} \\
z_{it} &= \zeta_{it}\sqrt{w_{it}} \\
\sigma_{it} &= exp(v_{it}/2) \\
v_{it} &= \phi_i v_{it-1} + \sigma_{i\epsilon}\epsilon_{it} \\
\epsilon_{it} &\sim n.i.i.d.(0,1)
\end{aligned}$$

where $\zeta_{it} \sim N(0,1)$ and $(p-2)w_{it}^{-1} \sim \chi_p^2$ independent of ζ_{it}.

The mean and variance of $\log z_{it}^2$ are known to be $-1.27+\psi(p/2)-\log(p/2)$ and $\pi^2/2+\psi(p/2)$ respectively, where $\psi(.)$ is the digamma function.

2.3 Stochastic volatility models with leverage effect

The independent SV models considered in previous subsections do not allow volatility to depend on the direction of price changes: so-called leverage effect (the volatility of returns tends to increase when the price drops). Black (1976) and Christie (1982) have found empirical evidence of this effect. Several different SV models have been employed in the literature for empirically

[4]Evidence in favor of fat-tails has been uncovered by Gallant et al. (1997) and Geweke (1994).

assessing the leverage effect (see, for example, Asai and McAleer (2005), Jacquier et al. (2004) and Yu (2005)). Here we use the model with fat-tails and correlated errors. Such model, called SV-t with leverage, is given by

$$
\begin{aligned}
r_{it} &= \sigma_{it} z_{it} \\
z_{it} &= \zeta_{it}\sqrt{w_{it}} \\
\delta_{it} &\sim n.i.i.d.(0, 1-\rho^2) \\
\zeta_{it} &= \delta_{it} + \rho\epsilon_{it} \\
\sigma_{it} &= exp(v_{it}/2) \\
v_{it} &= \phi_i v_{it-1} + \sigma_{i\epsilon}\epsilon_{it} \\
\epsilon_{it} &\sim n.i.i.d.(0, 1)
\end{aligned}
$$

where $\zeta_{it} \sim N(0, 1)$ and $(p-2)w_{it}^{-1} \sim \chi_p^2$ independent of ζ_{it}.

Since $corr(z_{it}, \epsilon_{it}) = \rho$, we have the leverage effect when $-1 < \rho < 0$.

3 A new proxy for market volatility

Now, we assume that the basic model (M1) holds and that the log-volatilities, $\log\sigma_{it}^2$, can be represented as the sum of two unobservable mutually orthogonal components, a common component, χ_{it}, driven by few (fewer than n) common factors, and an idiosyncratic component, ξ_{it}, driven by n idiosyncratic factors, that is

$$\log\sigma_{it}^2 = \chi_{it} + \xi_{it} = \sum_{j=1}^{q} b_{ij}(L)u_{jt} + \xi_{it}$$

where L stands for lag operator and the filters $b_{ij}(L)$ are one-sided in L and their coefficients are square summable. The q common shocks $(u_{jt};\ j = 1, ..., q;\ t \in Z)$ are assumed to be mutually orthogonal white noise processes (at all leads and lags) with unit variance. It important to note that the idiosyncratic components are not assumed mutually orthogonal across the market.

We propose the following proxy for the average volatility

$$\widehat{\sigma}_t^2 = \sum w_{it}\exp(\widehat{\chi}_{it})$$

where $\widehat{\chi}_{it}$ is the estimator of common component χ_{it}.

However, this estimator appears infeasible as $\log\sigma_{it}^2$ is not observable.

In order to solve this problem, we proceed by considering the innovation process η_t defined by

$$\eta_{it} = \log z_{it}^2 - E(\log z_{it}^2)$$

Since $E(\log z_{it}^2) = -1.27$, then

$$\log z_{it}^2 = \eta_{it} - 1.27$$

Thus we have that

$$\log r_{it}^2 = \log \sigma_{it}^2 + \eta_{it} - 1.27 = \chi_{it} + \xi_{it} + \eta_t - 1.27$$

or

$$\log r_{it}^2 + 1.27 = \chi_{it} + \xi_{it} + \eta_{it}$$

Now, we note that

$$\lambda_{it} = \xi_{it} + \eta_{it}$$

satisfies every property that a idiosyncratic component has to have, that is $\{(\lambda_{1t}, \lambda_{2t}, ..., \lambda_{nt})';\ t \in Z\}$ is a zero-mean stationary vector process for any n and $E(\lambda_{it} u_{j,t-k}) = 0$ for any i, j, t, and k. So, we can consider λ_{it} the idiosyncratic component of $\log r_{it}^2 + 1.27$. It follows that $\log \sigma_{it}^2$ e $\log r_{it}^2 + 1.27$ have the same common component. Thus, the common component of $\log \sigma_{it}^2$ can be estimated using the observations $\log r_{it}^2 + 1.27$, by applying the method based on the generalized dynamic factors models proposed in Forni et al. (2000, 2001, 2004).[5]

Of course, we can not pretend that such an estimator works as well as that we could obtain if $\log \sigma_{it}^2$ is observable. The latter is better since, though $\log \sigma_{it}^2$ and $\log r_{it}^2 + 1.27$ have the same common component, $\log r_{it}^2 + 1.27$ has an idiosyncratic component $\lambda_{it} = \xi_{it} + \eta_{it}$ heavier than $\log \sigma_{it}^2$. On the basis of our simulations, it is confirmed this conclusion though the performance of $\widehat{\chi}_{it}$ is not at all bad.

Now, we assume that the model (M2) or (M3) holds. In this framework the proxy of the average volatility obtained from common components is again:

$$\widehat{\sigma}_t^2 = \sum w_t \exp(\widehat{\chi}_{it})$$

where the common components are estimated on the basis of the observations $\log r_{it}^2 + 1.27 - \psi(p/2) + \log(p/2)$. In fact, under the hypothesis that z_{it} has a t distribution with $p > 3$ degrees-of-freedom, the common component of the log-volatility coincides with that of $\log r_{it}^2 + 1.27 - \psi(p/2) + \log(p/2)$. In general, the common component of the $\log \sigma_{it}^2$ coincides with the common component of the $\log r_{it}^2 + k$ where the value of constant k depends on the distribution of z_{it}.

[5] In particular, by applying this method, the common component is (non-parametrically) consistently estimated as both the size n of the cross-section and the series length T goes to infinity.

4 Simulation experiments

To assess the performance of the new proxy we have carried out Monte Carlo experiments on the three models M1, M2 and M3. In all models the values of the autoregressive parameter ϕ_i are uniformly distributed over [0.90, 0.995]. This choice is motivated by empirical studies: the typical values estimated from real returns data belong to this interval. Following Sandmann and Koopman (1998), for each value of ϕ_i, the values of $\sigma_{i,\epsilon}$ are selected so that the square of coefficient of variation:

$$CV_i^2 = \frac{var(\sigma_{it}^2)}{(E(\sigma_{it}^2))^2} = exp\left(\frac{\sigma_{i\epsilon}^2}{1-\phi_i^2}\right) - 1$$

takes the value of 0.5. High values of the ratio of volatility variance to its squared mean indicate pronounced relative strength of the stochastic volatility process while low values of CV signify that the model is close to the one of constant volatility. For the given values of ϕ_i, the $\sigma_{i,\epsilon}$ assume values around 0.2, coherently with the empirical evidence. In the models M2 and M3, the degrees-of-freedom (df) are posed equal to 7. The parameter estimates vary between 6 and 13 (see Liesenfeld and Jung (2000) and Bollerslev (1987)); we assume the lower value in order to emphasize the heaviness of the tails. In the model M3 the correlation coefficient (ρ) is posed equal to -0.3 (see Omori et al. (2005), Yu (2005)).

In all these models we have posed

$$\begin{aligned}\epsilon_{it} &= \sqrt{1-\gamma^2}u_t + \gamma e_{it} \\ u_t \sim n.i.i.d.(0,1), &\quad e_{it} \sim n.i.i.d.(0,1).\end{aligned}$$

where $0 < \gamma < 1$. It implies the data generating process is such that the log of volatility is expressible as sum of a common component (we assume only one factor) and an idiosyncratic component.

Indeed, we have that

$$\log \sigma_{it}^2 = v_{it}$$

On the other hand, we have that

$$v_{it} = \frac{\sigma_{i\epsilon}\sqrt{1-\gamma^2}}{1-\phi_i L}u_t + \frac{\sigma_{i\epsilon}\gamma}{1-\phi_i L}e_{it}$$

So

$$\log \sigma_{it}^2 = \frac{\sigma_{i\epsilon}\sqrt{1-\gamma^2}}{1-\phi_i L}u_t + \frac{\sigma_{i\epsilon}\gamma}{1-\phi_i L}e_{it}$$

The common component is given by

$$\chi_{it} = \frac{\sigma_{i\epsilon}\sqrt{1-\gamma^2}}{1-\phi_i L} u_t$$

and that idiosyncratic by

$$\xi_{it} = \frac{\sigma_{i\epsilon}\gamma}{1-\phi_i L} e_{it}.$$

We show two sets of results for $\gamma = 1/\sqrt{2}$ and $\gamma = 0.95$. The former value of γ implies that the variance of the common component is equal to that of idiosyncratic one, whereas the latter implies a bigger variance of the idiosyncratic component.

We generated data from each model with $n = 50, 100$ and 200 and $T = 200, 500, 1000$ and 2000. Each experiment was replicated 1,000 times. For simplicity we impose that the weights w_{it} equal $1/n$. In the experiments we compared our proxy, $\widehat{\sigma}_t^2$, with the mean of squared returns of asset in the basket, that is

$$\overline{r_t^2} = \frac{1}{n} \sum r_{it}^2$$

We measured the performance of the proxies by means of the relative mean square errors, defined respectively by:

$$RMSE_{\widehat{\sigma}_t^2} = \frac{\sum_{t=1}^{T} (\widehat{\sigma}_t^2 - \sigma_t^2)^2}{\sum_{t=1}^{T} (\sigma_t^2)^2}$$

and

$$RMSE_{\overline{r_t^2}} = \frac{\sum_{t=1}^{T} (\overline{r_t^2} - \sigma_t^2)^2}{\sum_{t=1}^{T} (\sigma_t^2)^2}.$$

Tables 1 and 2 present the simulation results.

A first notable finding is that the performance of $\widehat{\sigma}_t^2$ betters as the number of the series n and the time dimension T in the basket increase.

As for the model M1, that assumes the normality of shocks of the returns, the results show that our proxy $\widehat{\sigma}_t^2$ underperforms systematically $\overline{r_t^2}$ when the variance of the common component is equal to that of idiosyncratic one ($\gamma = 1/\sqrt{2}$) whereas it does not happen when $\gamma = 0.95$. As result from the Table 2, our proxy outperforms for $n = 50$.

In presence of fat tails (M2) and assuming again ($\gamma = 1/\sqrt{2}$), $\widehat{\sigma}_t^2$ outperforms $\overline{r_t^2}$ beginning from $t = 1000$ and until $n = 100$. When we assume fat tails with leverage effects (M3), our proxy outperforms for $t = 500$ and for $n = 50$, though the relative advantage decreases as n increases.

Table 1: Average of relative mean square error across 1,000 replications. ($\gamma = 1/\sqrt{2}$)

	n=50		n=100		n=200	
	$\widehat{\sigma}_t^2$	$\overline{r_t^2}$	$\widehat{\sigma}_t^2$	$\overline{r_t^2}$	$\widehat{\sigma}_t^2$	$\overline{r_t^2}$
Model M1						
$T = 200$	0.1701	0.0494	0.1054	$0,0247$	0.0784	0.0123
$T = 500$	0.0821	0.0492	0.0510	0.0248	0.0375	0.0125
$T = 1000$	0.0599	0.0492	0.0395	0.0249	0.0307	0.0124
$T = 2000$	0.0572	0.0497	0.0358	0.0250	0.0261	0.0125
Model M2						
$T = 200$	0.1905	0.0859	0.1154	0.0431	0.0817	0.0218
$T = 500$	0.0892	0.0865	0.0517	0.0431	0.0388	0.0217
$T = 1000$	0.0640	0.0869	0.0404	0.0436	0.0311	0.0219
$T = 2000$	0.0600	0.0868	0.0367	0.0436	0.0257	0.0218
Model M3						
$T = 200$	0.1901	0.1058	0.1140	0.0551	0.0896	0.0305
$T = 500$	0.0888	0.1076	0.0575	0.0564	0.0402	0.0310
$T = 1000$	0.0690	0.1086	0.0441	0.0568	0.0340	0.0311
$T = 2000$	0.0634	0.1082	0.0397	0.0564	0.0290	0.0312

Table 2: Average of relative mean square error across 1.000 replications. ($\gamma = 0.95$)

	n=50		n=100		n=200	
	$\widehat{\sigma}_t^2$	$\overline{r_t^2}$	$\widehat{\sigma}_t^2$	$\overline{r_t^2}$	$\widehat{\sigma}_t^2$	$\overline{r_t^2}$
Model M1						
$T = 200$	0.0837	0.0567	0.0472	0.0285	0.0353	0.0143
$T = 500$	0.0472	0.0573	0.0321	0.0288	0.0274	0.0144
$T = 1000$	0.0387	0.0573	0.0286	0.0288	0.0241	0.0144
$T = 2000$	0.0324	0.0572	0.0300	0.0288	0.0260	0.0144
Model M2						
$T = 200$	0.0938	0.1128	0.0514	0.0585	0.0362	0.0290
$T = 500$	0.0507	0.1133	0.0333	0.0578	0.0271	0.0290
$T = 1000$	0.0402	0.1131	0.0283	0.0579	0.0241	0.0289
$T = 2000$	0.0334	0.1139	0.0255	0.0574	0.0236	0.0288
Model M3						
$T = 200$	0.0954	0.1196	0.0513	0.0591	0.0393	0.0299
$T = 500$	0.0506	0.1188	0.0334	0.0598	0.0267	0.0302
$T = 1000$	0.0400	0.1166	0.0284	0.0601	0.0241	0.0306
$T = 2000$	0.0336	0.1187	0.0258	0.0594	0.0237	0.0299

The outperformance of $\widehat{\sigma}_t^2$ is clear when the variance of the idiosyncratic component is bigger than that of the common one (see Table 2).

Overall, the above simulations results suggest that the proxy $\widehat{\sigma}_t^2$ is suitable to be used to estimate volatility when the conditional density function of returns exhibits leptokurtosis and skewness. This is an important point because these features appear to be typical for the financial time series.

5 Conclusions

In this paper we have introduced a new method to compute a daily measure of volatility. The main feature of this method is that it is based upon the estimate of the common components of the volatilities of a basket of returns. Using simulated time series, we illustrated that this method performs better than the an average of squared daily returns in measuring volatility.

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