E C O N O M I C S B U L L E T I N

The Edgeworth Conjecture in a Public Goods Economy: An Elementary Example

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Abstract

We show by a simple example that in a public goods economy consisting of identical individuals with symmetric Cobb-Douglas preferences the core of the economy does not con-verge to the Lindahl solution when the number of agents goes to infinity. This confirms in an elementary way that the Edgeworth conjecture does not necessarily hold in an economy with a public good.

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The Edgeworth Conjecture in a Public Goods Economy

An Elementary Example

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Abstract: We show by a simple example that in a public goods economy consisting of identical individuals with symmetric Cobb-Douglas preferences the core of the economy does not converge to the Lindahl solution when the number of agents goes to infinity. This confirms in an elementary way that the Edgeworth conjecture does not necessarily hold in an economy with a public good.

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1. Introduction

The Lindahl equilibrium is the most prominent cooperative solution in a public goods economy. Being based on price-taking behaviour of the agents it clearly parallels the competitive allocation in a private goods economy. In addition, there is a further analogy with the private goods case since the Lindahl solution belongs to the core of the economy, which means that no subgroup of agents can attain a Pareto improvement by separating and providing the public good by using their own endowments alone (see Foley, 1970). But unlike a market economy, in which the core converges to the competitive equilibrium, the core of a public goods economy (for a general description see Weber and Wiesmeth, 1990) does not necessarily shrink to the Lindahl equilibrium if the original economy is replicated infinitely often, i.e. the "Edgeworth conjecture" does not hold in the public goods case. This important result, which robs the Lindahl solution some of its distinctiveness among Pareto optimal allocations, has been shown by Muench (1972), Milleron (1972, pp. 460-483), Champsaur et al. (1975), or Ellickson (1978) by rather intricate examples which indeed give the impression that it "is difficult to provide an accessible formal demonstration" (Cornes and Sandler, 1996, p. 303) for the non-convergence of the core in a public goods economy. In this note we, however, want to show that it is possible to provide an elementary example that, like Milleron's (1972) more sophisticated example, is based on identical Cobb-Douglas preferences and a linear technology for producing the public good.

2. The Example

Consider an economy consisting of $2n$ individuals $i = 1,...,2n$ that all have the same income $y_i = 1$ and the same Cobb-Douglas utility function $u(x_i, G) = x_i G$, where x_i denotes individual i's private consumption and G is public good supply. The public good is produced by a constant returns to scale economy for which the marginal rate of transformation between the private and the public good is normalised to one. The aggregate budget constraint therefore is

(1) $$\sum_{i=1}^{2n} x_i + G = \sum_{i=1}^{2n} y_i = 2n.$$

In this situation the (symmetric) Lindahl solution $L(n) = (x_1^L(n),...,x_{2n}^L(n), G^L(n))$ is, for any $n \in \mathbb{N}$, given by public good supply $G^L(n) = \sum_{i=1}^{2n} y_i / 2 = n$ and the private consumption levels $x_i^L(n) = y_i / 2 = 1/2$ for all agents $i = 1,...,2n$.

For an arbitrary n we now consider another Pareto optimal allocation $A(n) = (x_1^A(n),...,x_{2n}^A(n), G^A(n))$, in which public good supply clearly must be $G^A(n) = n$ $= G^L(n)$ but the public good contributions differ between two groups of individuals of equal size n, the high and the low contributors. We assume that in each $A(n)$ the public good contribution of a high contributor is $\frac{5}{8}$ and that of a low contributor only is $\frac{3}{8}$. Thus, independent of n, private consumption in $A(n)$ is $x_h^A(n) = 3/8$ for a high and $x_l^A(n) = 5/8$ for a low contributor, which gives utility levels $u_h^A(n) = \frac{3}{8}n$ and $u_l^A(n) = \frac{5}{8}n$ for high and low contributors, respectively.

In order to show that each allocation $A(n)$ is in the core of the economy we assume that a coalition consisting of $k \geq 0$ high contributors and $m \geq 0$ low contributors leaves the allocation $A(n)$ and provides the public good solely by use of its own income, which in total is $k + m$. In an efficient standalone allocation $S(k,m)$ of this separating group public good supply is $G^S(k,m) = \frac{k+m}{2}$ which is smaller than public good supply $G^A(n) = n$ in $A(n)$. Therefore, if a low contributor is not to be worse off in $S(k,m)$ as compared to $A(n)$ she needs a private consumption level that is higher than $x_l^A(n) = 5/8$. Then, however, there must be at least some high contributor j for whom private consumption in $S(k,m)$ is below

(2) $$\hat{x}_h^S(k,m) := \frac{1}{k}\left(\frac{k+m}{2} - \frac{5}{8}m\right) = \frac{1}{2} - \frac{1}{8}\frac{m}{k}.$$

Otherwise the budget constraint (1) would be violated. Thus, the following estimate for agent j's utility $u_j^S(k,m)$ in $S(k,m)$ can be given:

(3) $$u_j^S(k,m) \leq \hat{x}_h^S(k,m) \cdot G^S(k,m) = \left(\frac{1}{2} - \frac{1}{8}\frac{m}{k}\right)\left(\frac{k+m}{2}\right) <$$
$$< \left(\frac{1}{2} - \frac{1}{8}\frac{m}{n}\right)\left(\frac{n+m}{2}\right) = \frac{1}{16n}(4n-m)(n+m)$$
$$< \frac{1}{16n}(6n^2 - 2(m-n)^2) < \frac{6n^2}{16n} = \frac{3}{8}n = u_h^A(n).$$

This shows that at least one member of the coalition must be worse off after the deviation which means that the coalition cannot block $A(n)$. Since this argument holds for an arbitrary coalition, $A(n)$ is in the core independent of the size of n. Obviously, the allocation $A(n)$ is quite distinct from the Lindahl solution for all n which confirms that the core of the specific public good economy considered here does not converge to the Lindahl equilibrium when n goes to infinity.

This example can be easily generalized in the following way: Assume again that, in the identical situation as considered in the previous example, there are again two groups of low and high contributors each consisting of n individuals. In the original allocation $A(n)$ private consumption of a high contributor $x_h^A(n)$ now, however, is allowed to be some arbitrary $z \in (0, \frac{1}{2})$, such that private consumption of a low contributor $x_l^A(n)$ becomes $1 - z \in (\frac{1}{2}, 1)$. By a calculation quite analogous to that in our numerical example it can be shown that, independent of the size of n, the allocation $A(n)$ is in the core if and only if $z \geq \frac{1}{4}$, i.e. the chance for finding a coalition that is able to block the original allocation is increased when the distribution of private consumption becomes more skewed. This generalization also shows that the core of a public goods economy may contain a great many of allocations even if the size of the economy goes to infinity.

3. Conclusion

In the Cobb-Douglas framework it is also possible to provide other counterexamples to the Edgeworth conjecture that also could be treated as not too demanding exercises in an intermediate Public Finance course. So, if an economy consisting of two agents with the same income levels but different Cobb-Douglas preferences is replicated it turns out that public good supply in the Lindahl solution and the Moulin egalitarian-equivalent solution (see Moulin, 1987) may diverge when the number of replications is increased (see Buchholz and Peters for the calculations in this case). But as it is known as a general fact, that not only the Lindahl solution but also the Moulin solution always is an element of the core this observation implies as well that the core of the economy does not shrink to the Lindahl solution when the size of the economy goes to infinity. But note that, even though the Edgeworth conjecture is not valid in many simple cases it nevertheless may hold true also in a public goods economy when specific assumptions on preferences are additionally imposed (see Conley, 1994, for such positive results on core convergence in a public goods economy).

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