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On the Existence and uniqueness of Price Equilibrium with Multi-Store Firms

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Abstract

The paper gives a proof of the existence and the uniqueness of price equilibrium in a multi-product, multi-firm competition framework. It illustrates how the price levels depends on the inter-brand portfolio substitution and on the intra-brand portfolio substitution

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1. Introduction

The paper gives a proof of the existence and the uniqueness of price equilibrium in a multi-product, multi-firm competition framework. In fact, the impact of store location of a multi-store firm, or that of its competitors, on the price setting strategies of firms, is a subject that has been little explored in the industrial organization literature[1]. The standard models of horizontal differentiation (e.g. Hotelling, 1929, Salop, 1979) constitute an appropriate framework to treat this question. However, not many models have retained the original structure to study pricing strategies in the context of *multi-store* competition[2]. Levy and Reitz (1992) analyze the pricing strategies of firms in a multi-product framework, in which one firm controlling two stores is in competition with two mono-store competitors, using the circular model of Salop (1979). More recently, Giraud-Héraud et al. (2003) have studied price settings in the context of a single multi-product firm interacting with several mono-product competitors. The present paper generalizes these two contributions by identifying the unique price equilibrium that prevails in a complex context whatever the number of multi-product firms in competition, the number of stores owned by each firm, or the location of these stores. The main properties of the equilibrium that leads to a spatial differentiated pricing over the product line of each firm are also characterized.0

The paper is organized as follows. Section 2 presents our model. Section 3 contains our main existence and uniqueness results, and equilibrium properties. Section 4 concludes.

2. Bertrand-Nash equilibrium with multi-store firms

Consider a standard circular model of spatial differentiation, generalized for multi-store (or multi-product) firms. There are Q firms, F_q $(q = 1,\dots,Q)$, and each firm F_q owns n_q stores. We denote by $N=\sum_{q=1}^{Q} n_q$ the total number of stores, and we assume that all stores (denoted by $i=1,\dots,N$) are spread over a circle of perimeter d. Each store i sells the same product, but in different locations. *Let* x_i denote the location of store i, that is the curvilinear abscissa $x_i \in [0,d]$ with $x_i \neq x_j$ if $i \neq j$. Store i sells the product at price p_i $(i=1,\dots,N)$ with zero (marginal) production cost.
There is a continuum of consumer types θ distributed with unit density over $[0,d]$. The transportation cost incurred by consumer of type θ, when he consumes product x_i rather than θ, is given by $C(|\theta - x_i|)$ where $|\theta - x_i|$ denotes the geodesic distance between θ and i.

Figure 1 below illustrates the model. In this figure, firm F1 holds stores $i = 1,2,3,4$; firm F2 holds stores $i = 5$; Firm F3 holds stores i = 6, $i = 7$; etc. Note that, in our general model, firms may, or may not, hold stores with adjacent locations.

[1] The terms “store” and “brand” are used equivalently in this paper.

[2] A number of articles have explored this question modifying certain assumptions of these models, either with respect to the preferences of consumers (Klemperer, 1992 , Janssen et al., 2005) or with respect to strategic conjectures on the nature of competition (e.g. Cournot competition in Debashis et al., 2002). Thus, the arbitrage of the consumer between the costs of covering the distance to buy a commodity and its price, a fundamental assumption in the models of Hotelling (1929) and Salop (1979), does not figure in these papers.

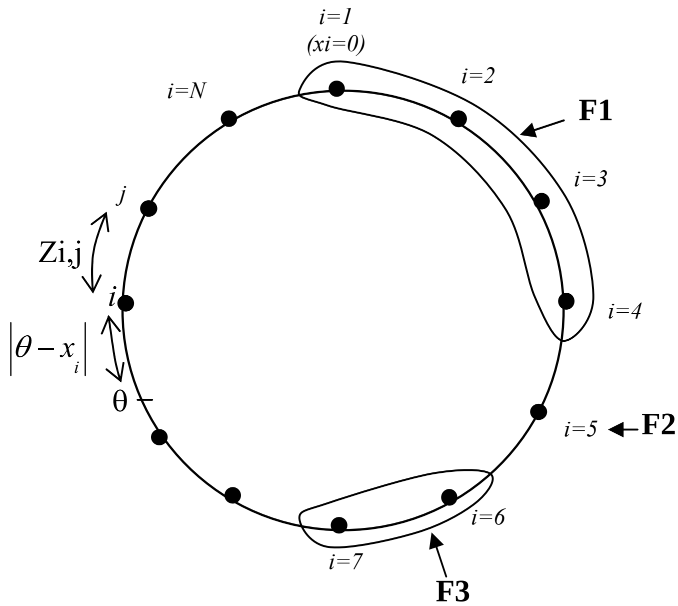

Figure 1 : Circular model with multi-store firms

Let r denote the reservation price common to all consumers. Consumers are allowed to buy one unit of a differentiated good, so that a consumer of type θ purchasing a product x_i at price p_i obtains a utility of:

$$U_\theta(r,x_i,p_i)=r-p_i-C(|\theta-x_i|) \quad ; \tag{1}$$

$$C(|\theta-x_i|)=a\,(\theta-x_i)^2 \quad (a>0) \tag{2}$$

The consumer's product choice problem can then be written as:

$$\underset{(x_i)_{i=1,..,N}}{Min}\ \{p_i + a\,(\theta - x_i)^2\} \tag{3}$$

Let $Z_{i,j}$ be the shortest geodesic distance between two stores i and j (for $j > i$ $Z_{i,j}=Z_{j,i}=inf\{(x_j-x_i),(d-x_j+x_i)\}$). For two adjacent stores i and i-1, let $\lambda_i=\frac{1}{Z_{i-1,i}}$, for $i = 1,\ldots,N$ with $Z_{0,1} = Z_{1,N}$.

For any market price vector, $p = (p_1,\ldots,p_N)$, let $D_i(p)$ and $B_i(p) = p_i\,D_i(p)$ denote the demand and the profit of a store i respectively. By determining the location of the consumers who are indifferent between buying from store i-1 or i, and between store i and $i+1$, the demand faced by store i can be expressed as follows:

$$D_i(p)=\frac{1}{2ad}[-(\lambda_i+\lambda_{i+1})p_i+(\lambda_i p_{i-1}+\lambda_{i+1}p_{i+1})+a(\frac{1}{\lambda_i}+\frac{1}{\lambda_{i+1}})] \qquad i=1,\ldots,N\,. \tag{4}$$

Since the demand $D_i(p)$ depends only on p_i, p_{i-1} and p_{i+1} among all the components of price vector p, it is useful to distinguish all the adjacent stores owned by a same firm F_q in order to investigate the Bertrand-Nash equilibrium between the Q firms. In order to do this we introduce some definitions.

A *k*-partition of stores: This is a partition of the N stores into K ranges $N=\sum_{k=1}^{K} n_k$.

Range *k* of firm *i*, $\Re_k$: $\Re_k$ is the kth set of neighboring brands belonging to the same firm i such that only two of these brands (the peripheral brand of the range) are directly exposed to brands owned by other firms.

The central store(s) of the range $\Re_k$, $m(\Re_k)$: $m(\Re_k)=(j-1)+\frac{n_k}{2}$ for even values of n_k (in this case there are two central stores) and $m(\Re_k)=(j-1)+\frac{n_k+1}{2}$ for uneven values of n_k (in this case there is only one central store).

Degree of exposure to competition of store $i \in \Re_k$, $d(i)$:

$$d(i)=\begin{cases} i-j & \text{if } i \le m(\Re_k) \\ \\ n_k+j-i & \text{if } i \ge m(\Re_k) \end{cases} \tag{5}$$

Peripheral stores of the range $\Re_k$: The stores i , $i=1,\ldots,N$ such that $d(i) = 0$.

Finally, let $\pi_{\Re_k}(p)=\sum_{i\in\Re_k} B_i(p)$ denote the profit of a range $\Re_k$ and $\pi_{F_q}(p)=\sum_{i\in F_q} B_i(p)=\sum_{\Re_k\subset F_q}\pi_{\Re_k}(p)$ denote the profit of a firm F_q.

From the above, it is clear that the Bertrand-Nash equilibrium between the Q firms can be analyzed through the Bertrand-Nash equilibrium between the K ranges. Each firm is indifferent about the number of ranges possessed by the other firms. With the assumption that all of products have positive market shares, the best response for each range $\Re_k=\{j,\ldots,j+n_k-1\}$ only depends on the two prices of its neighboring stores, namely p_{j-1} and p_{j+n_k} . Moreover, as will be shown, the equilibrium prices of each store will depend only on its exposure to competition.

Recall that a Bertrand-Nash equilibrium is defined by a N component vector of prices $p^*=(p_1^*,\ldots,p_N^*)$ such that:

$$\forall F_q \quad \forall p\in\{p=(p_1,\ldots,p_N)\,/\,p_i=p_i^*, i\notin F_q\} \quad \pi_{F_q}(p^*)\ge\pi_{F_q}(p) \tag{6}$$

Then we have the following proposition.

Proposition When $(x_1,\ldots,x_n)$ are fixed, there exists a unique price equilibrium with a non-zero market share for each store.

Proof **:** Given that store *i*'s profit only depends on its own pricing and on the prices of its neighbors, the best response for the ranges can be isolated. For $\Re_k = \{j,j+1,...,j+n_k-1\}$ we can show that $\pi_{\Re_k}$ is concave in $(p_j,\ldots,p_{j+n_k-1})$, since the Hessian matrix of $\pi_{\Re_k}$ is a constant equal to $\nabla^2(\pi_{\Re_k})$:

$$\nabla^2(\pi_{\Re_k})=\frac{1}{ad}\begin{pmatrix} -(\lambda_j+\lambda_{j+1})\lambda_{j+1} & & & 0 \\ \lambda_{j+1} & & & \\ & \lambda_i \; -(\lambda_i+\lambda_{i+1}) \; \lambda_{i+1} & & \\ & & & \lambda_{j+n_k-1} \\ 0 & & & \lambda_{j+n_k-1} -(\lambda_{j+n_k-1}+\lambda_{j+n_k}) \end{pmatrix} \tag{7}$$

Let Δ_m ($m = 1,...,n_k$) denote the principal minor of $\nabla^2(\pi_{\Re_k})$ of order m. We have $\Delta_1=-(\lambda_j+\lambda_{j+1})$ and $\Delta_m=-(\lambda_{m+j-1}+\lambda_{m+j})\Delta_{m-1}-\lambda^2_{m+j-1}\,\Delta_{m-2}$, for $m = 2,...,n_k$ where $\Delta_0=1$. Therefore, we can easily demonstrate by induction that $\Delta_m=-\lambda_{j+m}\,\Delta_{m-1}+(-1)^m\prod_{i=1}^{m}\lambda_{j+i-1}$ for $m = 1,...,n_k$.

For odd values of *m*, we have $\Delta_m < 0$ (since $\lambda_i > 0$ and $\Delta_0 > 0$) and for even values of *m*, we have $\Delta_m > 0$.Thus, the matrix $\nabla^2(\pi_{\Re_k})$ is negative definite.

Then we can write the first order conditions setting $\frac{\partial \pi_{\Re_k}}{\partial p_i} = 0 \; for \; i = j,\ldots, j+n_k-1$ and obtain the best response to the prices (p_{j-1},p_{j+n_k}) for each range $\Re_k$ as below:

$$\left|\begin{array}{l} -\frac{\lambda_j}{2}\,p_{j-1}+(\lambda_j+\lambda_{j+1})p_j-\lambda_{j+1}\,p_{j+1}=\frac{a}{2}\left(\frac{1}{\lambda_j}+\frac{1}{\lambda_{j+1}}\right) \\ -\lambda_i\,p_{i-1}+(\lambda_i+\lambda_{i+1})p_i-\lambda_{i+1}\,p_{i+1}=\frac{a}{2}\left(\frac{1}{\lambda_i}+\frac{1}{\lambda_{i+1}}\right) \qquad i=j+1,\ldots,j+n_k-2 \\ -\lambda_{j+n_k-1}\,p_{j+n_k-2}+(\lambda_{j+n_k-1}+\lambda_{j+n_k})p_{j+n_k-1}-\frac{\lambda_{j+n_k}}{2}\,p_{j+n_k}=\frac{a}{2}\left(\frac{1}{\lambda_{j+n_k-1}}+\frac{1}{\lambda_{j+n_k}}\right) \end{array}\right. \tag{8}$$

Then from (8), all the first-order conditions for the set of ranges, $\Re_k$; $k = 1,...,K$ can be summarized in the standard form $Ap = B$ where *A* and *B* are matrices and *p* is a vector. We have $B \geq 0$. since $\lambda_i > 0$ for all *i*. Applying McKenzie's theorem (1959 theorem 4, p. 50) a necessary and sufficient condition for $Ap = B$ to have a unique solution $p \geq 0$ is that $A = (a_{i,h})_{i=1,..,N;h=1,..,N}$ has a quasi-dominant diagonal or the following properties are verified :

(P1) There exist $d_h > 0$ such that $d_h\,|a_{h,h}| \geq \sum_{i\neq h} d_i\,|a_{i,h}| \quad (h=1,\ldots,N)$

(P2) When $a_{i,h} = 0$, given $h\in H$ and $i\notin H$ for some set of indices *H*, the strict inequality holds for some $h\in H$.

We can verify the property *(P1)* taking $d_h = 1$ for all h, $h = 1,\dots,n$ and furthermore confirm that there is no set of indices H such that $a_{i,h} = 0$ given $h \in H$ and $i \notin H$. This completes the proof.

3. Properties

We describe now the main properties of the Bertrand-Nash equilibrium. Without loss of generality, we suppose the products are symmetrically differentiated inside each range :

$$\forall \Re_k=\{j,j+1,\dots,j+n_k-1\} \; \exists\, Z_k \; \forall i \in \Re_k-\{j+n_k-1\} \;\; Z_{i,i+1} \equiv Z_k \tag{9}$$

There is the same product differentiation between the members of each range, but the differentiation between the ranges (that is between the peripheral stores of two ranges possessed by two different firms) is not necessarily the same.

Let $\overline{Z}_k$ indicate the differentiation between the range $\Re_k = \{j,j+1,\dots,j+n_k-1\}$ and the range $\Re_{k+1}$ $(\overline{Z}_k = Z_{j+n_k-1,j+n_k})$. Therefore, the solution of system (8) takes the form $p_i = \alpha(i-j+1)^2+\beta(i-j+1)+\gamma$ with constant coefficients. Using (8) and after identification of the constant coefficients, we have the following formulation for the equilibrium prices:

$$\left|\begin{array}{ll} p_i^* = -\dfrac{aZ_k^2}{2}(i-j+1)^2 & \\ \quad + \dfrac{Z_k}{2}\left\{\dfrac{p^*_{j+n_k}-p^*_{j-1}}{(n_k-1)Z_k+\overline{Z}_{k-1}+\overline{Z}_k} + a[(n_k+1)Z_k+\overline{Z}_k-\overline{Z}_{k-1}]\right\}(i-j+1) & \\ \quad + \dfrac{1}{2}\left\{\dfrac{(\overline{Z}_{k-1}-Z_k)p^*_{j+n_k}+(\overline{Z}_k+n_kZ_k)p^*_{j-1}}{(n_k-1)Z_k+\overline{Z}_{k-1}+\overline{Z}_k} - a(\overline{Z}_k+n_kZ_k)(Z_k-\overline{Z}_{k-1})\right\} & \text{if } n_k>1 \\ & \\ p_j^* = \dfrac{1}{2(\overline{Z}_{k-1}+\overline{Z}_k)}(\overline{Z}_k\, p^*_{j-1} + \overline{Z}_{k-1}\, p^*_{j+1}) + \dfrac{a}{2}\overline{Z}_{k-1}\overline{Z}_k & \text{if } n_k=1 \end{array}\right. \tag{10}$$

Furthermore, using (4) and (8), it can be shown that the equilibrium market shares $D_i^* = D_i(p^*)$ verify :

$$\left|\begin{array}{l} \forall \Re_k, n_k>2 \;\; \forall i \in \Re_k\,, d(i)\neq 0 \quad D_i^* = \dfrac{Z_k}{2d} \\ \displaystyle\sum_{d(i)=0} D_i^* = \frac{1}{2} + \frac{1}{2d}\left[\sum_{k=1}^{K}(\overline{Z}_k+Z_k)\right] \end{array}\right. \tag{11}$$

In figures 2 and 3 we focus on a duopoly configuration with only one multi-product firm $F_1 = \Re_1 = \{1,\dots,n\}$ and a mono-product $F_2 = \Re_2 = \{N\}$ with $N = n+1$, in the market. We consider the case where any two stores of a firm are equidistant : $Z_1 = Z$. The distance between store 1 and N

is $\overline{Z}_2$ and the distance between n and N is $\overline{Z}_1$ and we have $(n-1)Z+\overline{Z}_1+\overline{Z}_2=d$.

Figures 2 and 3 show the kind of price and profit differentiation that could emerge in two polar cases: first, the symmetrical situation when $\overline{Z}_1 = \overline{Z}_2$ (figures 2a and 2b) and second, the asymmetrical situation, when $\overline{Z}_1$ is significantly greater than $\overline{Z}_2$ (figures 3a and 3b).

The variation of prices along the range depends on the degree of differentiation between the competitors and on the number of stores within the range. A multi-store firm chooses its brands' price vector as a function of its degree of exposure to outside competition. The closer a brand is to outside competition of another firm, the lower its price and vice versa.

In the symmetrical situation, the prices inside the range are increasing from the periphery to the center of the range. Peripheral stores offer the lowest prices while central stores protected from the external competition offer the highest prices. We show in figure 2b that for high values of $\overline{Z}_2$ and n, the highest profits in the range $\Re_1$ are those of the peripheral stores because of the greater market share that they capture. The price at store N is always below the prices at any store of the multi-store firm, but its profit is the highest in the industry, if its product is sufficiently differentiated.

For the asymmetrical situation, we suppose there is no differentiation between store *1* and store N, taking $\overline{Z}_2 = 0$, while store n is the most differentiated store from the rival N, taking $\overline{Z}_1 = d/2$. Figure 3a shows the variation of prices, which are increasing from store *1* to store n. Store n not only charges the highest price, but also enjoys the largest market share, thereby being the most profitable in the industry.

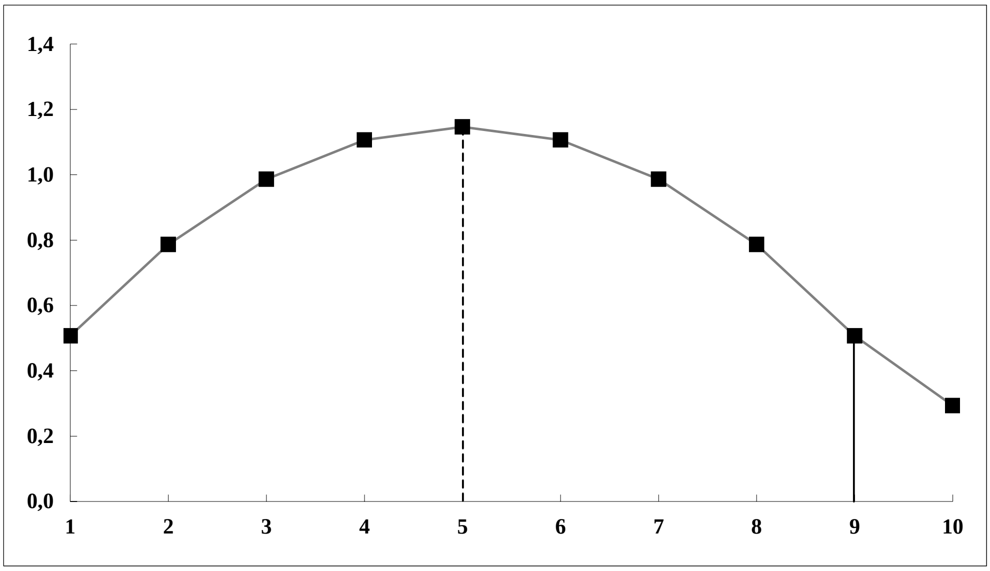

Figure 2a: Price Variation

Parameter configuration a = d = 2 and $\bar{Z}_1 = \bar{Z}_2 = d/10$; $n = 9$ $(N = 10)$

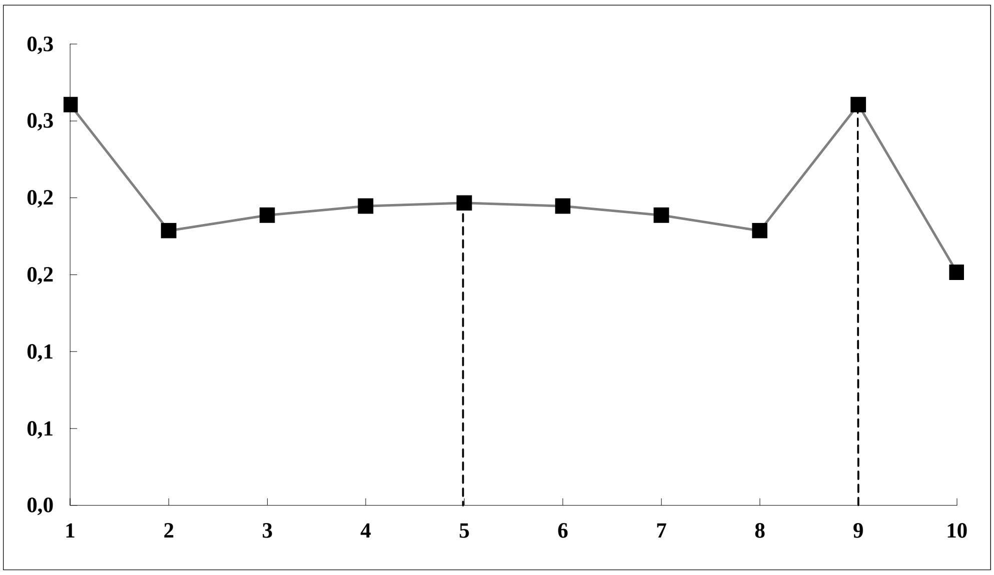

Figure 2b:Profit Variation

Parameter configuration a = d = 2 and $\bar{Z}_1 = \bar{Z}_2 = d/10$; $n = 9$ $(N = 10)$

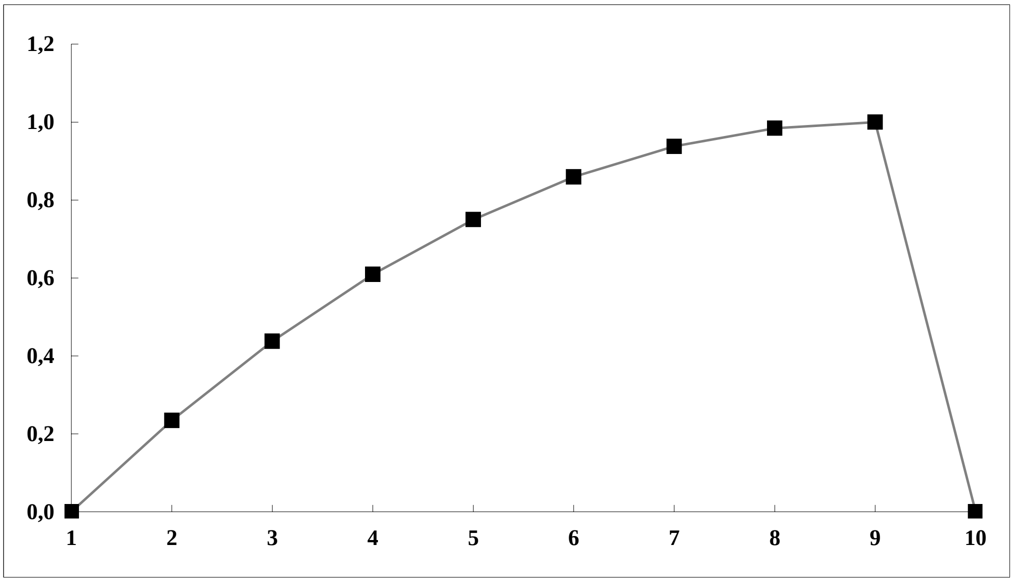

Figure 3a: Price Variation

Parameter configuration a = d = 2 and $\bar{Z}_1 = d/2$; $\bar{Z}_2 = 0$; $Z = d/16$ $n = 9$ $(N = 10)$

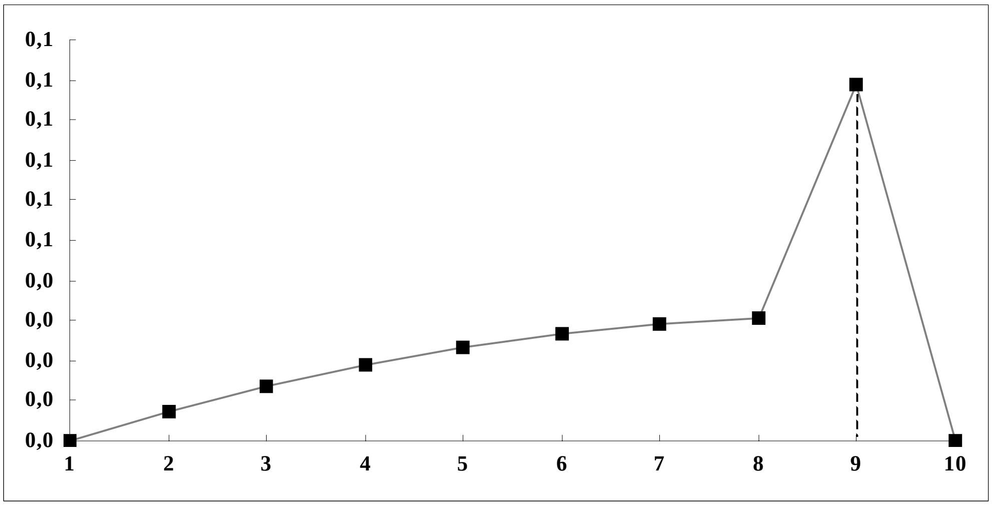

Figure 3b: Profit Variation

Parameter configuration a = d = 2 and $\bar{Z}_1 = d/2$; $\bar{Z}_2 = 0$; $Z = d/16$ $n = 9$ $(N = 10)$

4. Concluding Remarks

In this paper, we extended the standard duopoly model of spatial differentiation to a multi-store, multi-firm competition. We resolved the complex problem of identifying and confirming the existence and uniqueness of a price equilibrium for symmetric and asymmetric locations of brand portfolios of multi-product firms. The results obtained with multi-store firms are different from those obtained with single-store firms. The price variation over stores emerges as a function of the degree of the substitution of the portfolios of the different multi-product firms. Such results are in the line with some observations and empirical works (see for example Barron et al (2000) and Netz and Taylor (2002)). In many final product markets the multi-store phenomenon is gaining ground. Therefore, the proof of existence of an equilibrium in prices and its characterization contributes to a better understanding of the determinants of pricing strategies of firms in such markets, which in turn could be useful for the formulation of competition policy in the context of multi-store firm competition.

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