E C O N O M I C S B U L L E T I N

The Solow model in discrete time and decreasing population growth rate

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Abstract

This paper reformulates the neoclassical Solow-Swan model of economic growth in discrete time by introducing a generic population growth law that verifies the following properties: 1) population is strictly increasing and bounded; 2) the rate of growth of population is decreasing to zero as time tends to infinity. We show that in the long run the capital per worker of the model converges to the non-trivial steady state of the Solow--Swan model with zero labor growth rate. In addition we prove that the solutions of the model are asymptotically stable.

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1 Introduction

In the original neoclassical model of economic growth due to Solow (1956) and Swan (1956), it is assumed that labour force L grows at a constant rate $n > 0$. In discrete time it is natural to define this growth rate as:

$$n = \frac{L_{t+1} - L_t}{L_t} \tag{1}$$

which implies that

$$L_{t+1} = (1 + n) L_t, \tag{2}$$

Then the labour force growths exponentially, and for any initial level L_0, at time t the level of labour force is

$$L_t = L_0 (1 + n)^t. \tag{3}$$

This assumption is plausible only for small values of t because growing exponentially, labour force approaches infinity when t goes to infinity, which is clearly unrealistic. The simple Malthusian model can provide an adequate approximation to such growth only for an initial period but does not accommodate growth reductions due to competition for environmental resources such as food and habitat. Verhulst (1838) considered that a stable population would have a characteristic saturation level; this limit for the population size is usually called the *carrying capacity* of the environment[1]. To incorporate this numerical upper bound on the growth size Verhulst introduced the logistic equation as an extension of the exponential model.

It is well known that since the 1950s, population growth rate is decreasing and it is projected to decrease to zero during the next six decades. This decrease is particularly relevant in the group of developed countries but is also observable on a global scale. The decrease in the rate of growth is predominantly due to the aging of the population and, consequently, a dramatic increase in the number of deaths. From 2030 to 2050, the world population would grow more slowly than ever before in its history. (See Day (1996))

Then, as described in Maynard (1974), a more realistic law of growth of the labour force L_t must verify the following properties:

1. when population is small enough in proportion to environmental carrying capacity L_∞, then L grows at a constant rate $n > 0$,

2. when population is large enough in proportion to environmental carrying capacity L_∞, the economic resources become more scarce and this affect negatively population growth,

3. population growth rate is decreasing to 0.

In discrete time, the logistic equation due to Pielou (1969) and the Beverton-Holt equation (Beverton and Holt (1957)) are representative examples of population laws verifying these proprieties.[2].

[1]In Arrow et al (1995), Cohen (1995a), Cohen (1995b) and Daily and Ehrlich (1992) the reader can find detailed information about the concept of carrying capacity of human population.

[2]See Brianzoni el at (2007) and Cushing and Henson (2001).

In this paper we introduce a modification in the neoclassical economic growth model in discrete time by assuming that population growth follows the properties defined above. The same problem in continuous time was exhaustive studied (see Accinelli and Brida (2007), Brida (2008), Brida and Limas (2007), Donghan (1998), Guerrini (2006) and Mingari and Ritelli (2003)). The main contribution of this short paper is to study the proprieties of the model when time is represented in discrete form and to compare the results with the continuous model. Whenever one writes a dynamic model, a fundamental choice that has to be made is the question of discrete versus continuous time. This choice can affect dramatically outcomes of a model and conclusions that might draw from them, because dynamics of the two types of models can be completely different and lead to different predictions. This gives a first challenge to study a discrete version of the model. One of the main examples of a model that is rather different when we change timing is the logistic population law. In continuous time it is a simple model: all the solutions of the logistic differential equation converge monotonically to a constant level for any choice of the parameters of the model. From the other hand, the logistic difference equation can produce very complex and chaotic dynamics for a continuous range of its parameters. The complexity of the discrete logistic model has produced a line of research in the area of dynamical systems. After the publication of May (1976), the study of the mathematical properties of the logistic discrete equation produced a large number of economic applications of chaos theory (see Sordi (1996)) consisting of models with final equation that can be reduced to an equation of the logistic type. Most of these applications are traditional models revisited (see references in Sordi (1996)) and reformulated in a discrete version. In economic modeling both types of timing are present and there is not a common view between economists on which representation of time is better to model in economics. From an economic point of view, one can present several arguments in favor of discrete time: fundamental economic data is collected at discrete intervals, there are fundamental decisions made at discrete intervals, transformation of capital into investments depends on the length of a time lag, etc.[3] In the same way, there are several arguments in favor of the continuous representation of time: traditional science has used continuous timing, some economic variables are better represented in continuous time, the time lag in transformation of variables is so small that can be considered an instant, etc. From the mathematical perspective, in continuous time we model with differential equations while discrete dynamic models are represented by difference equations. This produces a technical difference because different tools must be used to study discrete and continuous models. Generically, when one changes a model from continuous to discrete time the complexity of dynamics is increased and this was our first challenge. Then, another challenge to study the discrete version of Brida (2008) is technical: we have a different mathematical object to study and then innovative techniques must be introduced.

Previous works about the traditional Solow model in discrete time are: Commendatore (2004), Day (1982), Schenk-Hoppé and Schmalfub (2001), Wei-Bin (2005) and (2007). We will resume the fundamental properties of this model to compare with our outcomes. The paper is organized as follows. In section 2 we review the classic Solow model with population constant growth rate and then we introduce the reformulate model and describes there qualitative properties. In particular, we show that economic growth accelerates when population growth rate decreases to zero and that

[3]See Licandro and Punch (2006) for a discussion of time dimension in economic models and references therein.

per capita capital converges to a constant value (that is independent of the population growth law and of the initial per capita capital) as time tends to infinity. Conclusions and future developments are summarized in the last section.

2 The Solow model with decreasing population growth rate.

The key elements of the original model are the production function, i.e. how the inputs of capital K and labour L are transformed into outputs, and how capital and labour change over time. In particular, the model assumes that:

1. the production function $F(K,L)$ satisfies the following conditions:
   (a) $F(\lambda K, \lambda L) = \lambda F(K,L)$, $\forall \lambda, K, L \in R^+$ (constant return to scale)
   (b) $F(K,0) = F(0,L) = 0$, $\forall K, L \in R^+$
   (c) $\frac{\partial F}{\partial K} > 0$, $\frac{\partial F}{\partial L} > 0$, $\frac{\partial^2 F}{\partial K^2} < 0$, $\frac{\partial^2 F}{\partial L^2} < 0$
   (d) $\lim_{K \to 0^+} \frac{\partial F}{\partial K} = \lim_{L \to 0^+} \frac{\partial F}{\partial L} = +\infty$; $\lim_{K \to +\infty} \frac{\partial F}{\partial K} = \lim_{L \to +\infty} \frac{\partial F}{\partial L} = 0$

2. the capital stock changes equal the gross investment $I = sF(K,L)$ minus the capital depreciation δK :
$$K_{t+1} - K_t = sF(K_t, L_t) - \delta K_t \tag{4}$$

3. the labour force L_t grows at a constant rate $n > 0$
$$L_{t+1} = (1+n) L_t \tag{5}$$

If $k = \frac{K}{L}$ is the capital per worker and $f(k) = F\left(\frac{K}{L}, 1\right) = F(k,1)$ is the production function in intensive form, we have the following proprieties:

$$f(0) = 0, \quad f'(k) > 0 \forall k \in R^+, \quad \lim_{k \to +\infty} f'(k) = 0, \quad \lim_{k \to 0^+} f'(k) = +\infty \quad \text{and} \quad f''(k) < 0 \forall k \in R^+ \tag{6}$$

From (4) and (5), we obtain the equation of motion for the model which describes how capital per worker varies over time:

$$k_{t+1} = \frac{s}{1+n} f(k_t) + \left(\frac{1-\delta}{1+n}\right) k_t \tag{7}$$

It is straightforward to prove that there exists a unique positive solution $\hat{k}_n$ of equation (7). The main properties of model (7) are:

1. $\forall k_0 > 0$, the solution $(k_t)_{t \in \mathbb{N}}$ of (7) with initial condition k_0 verifies: $\lim_{t \to +\infty} k_t = \hat{k}_n$

2. If $k_0 < \hat{k}_n$ then $(k_t)_{t\in\mathbb{N}}$ is strictly increasing and if $k_0 > \hat{k}_n$ then $(k_t)_{t\in\mathbb{N}}$ is strictly decreasing

3. $\hat{k}_n$ is asymptotically stable and it is a global attractor of (7)

4. $\hat{k}_n$ is a decreasing continuous function of n. In particular: $\lim\limits_{n\to 0^+} \hat{k}_n = \hat{k}$, where $\hat{k}$ is the positive equilibrium of (7) when $n = 0$. Then, if two economies have the same initial capital per worker, the economy with smaller rate of growth of population has bigger long run capital per worker.

5. If $k_0^1 < k_0^2$ then $k_t^1 < k_t^2, \forall t \in \mathbb{N}$. This means that if two economies have the same fundamentals, then the one with bigger initial capital per worker has bigger capital per worker for ever. Additionally, since $K_t = k_t L_t$, then we have that: $\lim\limits_{t\to+\infty} K_t = \infty$.

Now we modify the previous model by substituting equation (5) by a population law L_t such that:

1. $L_0 > 0; L_{t+1} > L_t, \forall t \geq 0$ and $\lim\limits_{t\to+\infty} L_t = L_\infty$
2. If $n_t = \frac{L_{t+1}-L_t}{L_t}$ then $n_{t+1} < n_t, \forall t \geq 0$ and $\lim\limits_{t\to+\infty} n(t) = 0$

Then the model is represented by the following new equation of motion describing how capital per worker varies over time:

$$k_{t+1} = \frac{s}{1+n_t} f(k_t) + \left(\frac{1-\delta}{1+n_t}\right) k_t \tag{8}$$

To solve it one needs a single initial condition k_0. Note that this is a **non autonomous** difference equation and then usual techniques to analyze stability (i.e. phase diagrams, eigenvalues) are not useful.

In the following lemmas we compare two solutions of (8) that have different initial conditions and differ on the population law.

Lemma 1 : *Given k_0^1 and k_0^2, such that $k_0^1 < k_0^2$, then the solutions $\left(k_t^1\right)_{t\in\mathbb{N}}$ and $\left(k_t^2\right)_{t\in\mathbb{N}}$ of (8) with initial conditions k_0^1 and k_0^2 respectively, verify that: $k_t^1 < k_t^2$ for all $t \in \mathbb{N}$.*

Proof: The result is obvious when $t = 0$. We now proceed by induction. Suppose that for a given $t \in \mathbb{N}$ the inequality $k_t^1 < k_t^2$ holds and we will show that the theorem holds for $t+1 \in \mathbb{N}$. Being that f is increasing we have that

$$k_{t+1}^1 = \frac{s}{1+n_t} f(k_t^1) + \left(\frac{1-\delta}{1+n_t}\right) k_t^1 < \frac{s}{1+n_t} f(k_t^2) + \left(\frac{1-\delta}{1+n_t}\right) k_t^2 = k_{t+1}^2 . \square \tag{9}$$

Remark: Note that this result is also true for the classical model and it implies that, if two economies have the same fundamentals, then the one with bigger initial capital per worker has bigger capital per worker for ever.

Lemma 2 *Given the equations*

$$k_{t+1} = \frac{s}{1+n_t^1} f(k_t) + \left(\frac{1-\delta}{1+n_t^1}\right) k_t$$

and

$$k_{t+1} = \frac{s}{1+n_t^2} f(k_t) + \left(\frac{1-\delta}{1+n_t^2}\right) k_t$$

If

$$k_0^1 = k_0^2.$$

and

$$n_t^1 \leq n_t^2 \textit{ for all } t \in \mathbb{N}$$

then

$$k_t^1 \geq k_t^2 \textit{ for all } t \in \mathbb{N}$$

Proof. The result is obvious when $t = 0$ and then the proof proceeds by induction. We suppose that for a given $t \in \mathbb{N}$, $k_t^1 \geq k_t^2$ holds and we will show that the lemma holds for $t+1 \in \mathbb{N}$:

$$\begin{aligned} k_{t+1}^1 &= \frac{s}{1+n_t^1} f(k_t^1) + \left(\frac{1-\delta}{1+n_t^1}\right) k_t^1 \geq \frac{s}{1+n_t^1} f(k_t^2) + \left(\frac{1-\delta}{1+n_t^1}\right) k_t^2 \geq \\ &\geq \frac{s}{1+n_t^2} f(k_t^2) + \left(\frac{1-\delta}{1+n_t^2}\right) k_t^2 = k_{t+1}^2 \end{aligned}$$

here we used that f is increasing and $n_t^1 \leq n_t^2$ for all $t \in \mathbb{N}$. ■

Remark: This proposition implies that for two economies with the same initial capital per worker, the economy with smaller rate of growth of population has bigger capital per worker for ever.

Theorem 1: If $(k_t)_{t\in\mathbb{N}}$ is a solution of $k_{t+1} = \frac{s}{1+n_t} f(k_t) + \left(\frac{1-\delta}{1+n_t}\right) k_t$, then:

$$\lim_{t\to+\infty} k_t = \hat{k}.$$

Proof:

Let $\varepsilon > 0$; we want to prove that there exists $H > 0$ such that $\forall t \geq H$, $\left|k_t - \hat{k}\right| < \varepsilon$.

From

$$\lim_{n\to 0^+} \hat{k}_n = \hat{k}$$

we know that there exist $\bar{n} > 0$ such that

$$\forall n \leq \bar{n}, \left|\hat{k}_n - \hat{k}\right| < \frac{\varepsilon}{3} \tag{10}$$

Let $t_1 \geq 0$ such that $n_{t_1} \leq \bar{n}$ and let k_t^1 and k_t^3 be the solutions of the difference equations

$$(A): \; k_{t+1} = \frac{s}{1+n_{t_1}} f(k_t) + \left(\frac{1-\delta}{1+n_{t_1}} \right) k_t \tag{11}$$

and

$$(C): k_{t+1} = sf(k_t) + (1-\delta) k_t \tag{12}$$

respectively with the initial condition

$$k_{t_1}^1 = k_{t_1}^3 = k_{t_1}. \tag{13}$$

Then the previous lemma implies that

$$k_t^1 \leq k_t \leq k_t^3, \forall t \in [t_1, +\infty). \tag{14}$$

We know that $\lim\limits_{t \to +\infty} k_t^3 = \hat{k}$ and then $\exists H_1 > 0$ such that

$$\forall t \geq H_1, \left| k_t^3 - \hat{k} \right| < \frac{\varepsilon}{3} \tag{15}$$

We also have that $\lim\limits_{t \to +\infty} k_t^1 = \hat{k}_{n_{t_1}}$ and then $\exists H_2 > 0$ such that

$$\forall t \geq H_2, \left| k_t^1 - \hat{k}_{n_{t_1}} \right| < \frac{\varepsilon}{3} \tag{16}$$

Thus, $\forall t \geq H = \max(H_1, H_2) \geq t_1$ it is:

$$\hat{k} - \frac{2\varepsilon}{3} < \hat{k}_{n_{t_1}} - \frac{\varepsilon}{3} < k_t^1 \leq k_t \leq k_t^3 < \hat{k} + \frac{\varepsilon}{3} \tag{17}$$

and this implies that

$$\left| k_t - \hat{k} \right| < \varepsilon, \forall t \geq H \blacktriangle \tag{18}$$

Remark: Note that $\hat{k}$ is not an equilibrium of (8). This proposition implies that there exists a constant (long run) value $\hat{k}$ that attracts any solution of the model as t tends to infinity. Additionally we have showed that the intrinsic rate of population growth n_t plays no role in determining the long run level of per capita output, because $\hat{k}$ is the unique positive solution of the equation $sf(k) = \delta k$ and it not depends on n_t.

Remark: It can be proved that the capital per worker of model (8) converges monotonically. In the appendix we show it and we present an alternative proof of Theorem 1.

Theorem 2: The solution of (8) with initial condition k_0 is asymptotically stable.

Proof: To prove the (Lyapunov) stability of $(k_t)_{t \in \mathbb{N}}$ (solution of (8) with initial condition k_0) we have to show that: $\forall \epsilon > 0, \exists \delta > 0$ such that for any solution $(q_t)_{t \in \mathbb{N}}$ of

$$k_{t+1} = \frac{s}{1+n_t} f(k_t) + \left(\frac{1-\delta}{1+n_t} \right) k_t$$

with initial condition q_0 verifying $|q_0 - k_0| < \delta$, then we have that

$$|k_t - q_t| < \epsilon, \forall t \in \mathbb{N}. \tag{19}$$

Let $\epsilon > 0$, and $(a_t)_{t\in\mathbb{N}}$ and $(b_t)_{t\in\mathbb{N}}$ the solutions of (8) with initial conditions $a_0 = k_0 - \epsilon$ and $b_0 = k_0 + \epsilon$ respectively. From the previous theorem we have that

$$\lim_{t\to+\infty} a_t = \lim_{t\to+\infty} b_t = \hat{k} \tag{20}$$

and then $\exists t_0 > 0$ such that $|a_t - k_t| < \epsilon$ and $|b_t - k_t| < \epsilon, \forall t \in [t_0, +\infty)$. Then, from lemma 1 we have that $\forall q_0 \in [a_0, b_0]$, if $(q_t)_{t\in\mathbb{N}}$ is the solution of (8) with initial condition q_0 we have that

$$a_t \leq q_t \leq b_t, \forall t \in [0, +\infty). \tag{21}$$

Thus, $\forall q_0 \in [a_0, b_0]$ the solution $(q_t)_{t\in\mathbb{N}}$ verifies

$$|k_t - q_t| < \epsilon, \forall t \in [t_0, +\infty). \tag{22}$$

Now we have to choose $0 < \delta < \epsilon$ such that for any solution $(q_t)_{t\in\mathbb{N}}$ of

$$k_{t+1} = \frac{s}{1+n_t} f(k_t) + \left(\frac{1-\delta}{1+n_t}\right) k_t$$

with initial condition q_0 verifying $|q_0 - k_0| < \delta$, then we have that

$$|k_t - q_t| < \epsilon, \forall t < t_0. \tag{23}$$

This is a consequence of the continuity of the functions

$$j_t(x) = \frac{s}{1+n_t} f(x) + \left(\frac{1-\delta}{1+n_t}\right) x$$

for $t = 1, 2, \ldots, t_0 - 1$. In particular we have that, given $\epsilon > 0$, $\exists \delta_{t_0} > 0$ such that $\forall x \in (k_{t_0-1} - \delta_{t_0}, k_{t_0-1} + \delta_{t_0})$ it is

$$|j_{t_0-1}(x) - j_{t_0-1}(k_{t_0-1})| < \epsilon$$

Then, for this $\delta_{t_0} > 0$, $\exists \delta_{t_0-1} > 0$ such that $\forall x \in (k_{t_0-2} - \delta_{t_0-1}, k_{t_0-2} + \delta_{t_0-1})$ it is

$$|j_{t_0-2}(x) - j_{t_0-2}(k_{t_0-2})| < \delta_{t_0}$$

and if we continue this procedure, we can find $\delta_1 > 0$ such that $\forall x \in (k_0 - \delta_1, k_0 + \delta_1)$ it is

$$|j_0(x) - j_0(k_0)| < \delta_2.$$

Any value $\delta < \min(\delta_1, \frac{\epsilon}{2})$ is the required value.

Then δ verifies that if $(q_t)_{t\in\mathbb{N}}$ is the solution of (8) with initial condition q_0 verifying $|q_0 - k_0| < \delta$, then it is

$$|k_t - q_t| < \epsilon, \forall t \in \mathbb{N}. \tag{24}$$

This shows that the solution of (8) is (Lyapunov) stable. From the previous theorem we have that for any solutions $(k_t)_{t\in\mathbb{N}}$ and $(q_t)_{t\in\mathbb{N}}$ of (8) it is

$$\lim_{t\to+\infty} k_t = \lim_{t\to+\infty} q_t = \hat{k} \tag{25}$$

and then it is

$$\lim_{t\to+\infty} [k_t - q_t] = 0. \tag{26}$$

This shows that the solution of (8) with initial condition k_0 is asymptotically stable.▲

Remark: This proposition implies that small variations of the initial per worker capital do not change very much the economic growth process.

Remark: We can analyze the impact of technology on economic growth through its impact on the environmental carrying capacity L_∞. We assume that technological development increases the carrying capacity of the environment, i.e. $L_\infty(A)$ is an increasing function of the variable A (the technology). Of course, if technology A just affects the environmental carrying capacity it does not impact the dynamics of the classical Solow model (with exponential population growth). However, it does impact the Solow model reformulated in this paper. As technological development leads to greater population, and as the steady state equilibrium values of consumption per capita, capital per capita and output per capita remain constant when technology develops, it implies that technology increases the aggregate levels of consumption, capital and output.

3 Conclusions

In growth theory it is usually assumed that population growth follows an exponential law. This is clearly unrealistic because, in particular, it implies that population goes to infinity when time goes to infinity. In this paper we suggest a more realistic approach by considering that population is strictly increasing and bounded, and that its rate of growth is strictly decreasing to zero. The paper shows that there exists a constant (long run) value $\hat{k}$ that attracts any solution of the model as t tends to infinity. Being $\hat{k}$ the unique positive solution of $sf(k) = \delta k$, it depends only on the technology f, the fraction s of output that is saved and the rate of capital depreciation δ, and thus the intrinsic rate of population growth n_t plays no role in determining the long run level of per capita output. Then two economies with different rate of growth of population (both of them decreasing to zero), but with the same technology f, fraction s of output that is saved and rate of capital depreciation δ, will converge to the same long run value $\hat{k}$. By the contrary, with exponential population growth an increase in the intrinsic rate of population growth leads to lower levels of long run output per capita. Moreover, since the previous theorem implies that $\hat{k}$ is a global attractor of equation (8), small variations of the initial per worker capital do not change very much the economic growth process. The paper also shows that long run values of per capita levels of consumption, capital and output are greater than those of the classical model. Thus, in the long run, economic growth is improved if labour force growth rate decreases. This is a motivation for policy makers to have an efficient population growth rate. Additionally, note that being $\lim_{t\to+\infty} L_t = L_\infty$ and $\lim_{t\to+\infty} k_t = \hat{k}$ then

$$\lim_{t\to+\infty} K_t < \infty$$

This is more realistic than in the original model where $\lim_{t\to+\infty} K_t = \infty$.

Finally, if we assume that $n_{t+1} < n_t, \forall t \geq 0$ and $\lim_{t\to+\infty} n(t) = \widetilde{n} > 0$, we can obtain the same results with suitable modifications. But this is material of future research.

4 Appendix

In this appendix we present a theorem to compare the solutions of the original Solow model and (8) when they start from the same initial condition. We show that the capital per worker of the reformulated model varies monotonically.

Theorem 3. Let $\left(k_t^1\right)_{t\in\mathbb{N}}$, $\left(k_t^2\right)_{t\in\mathbb{N}}$ and $\left(k_t^3\right)_{t\in\mathbb{N}}$ solutions of the following difference equations:

$$(A): \ k_{t+1} = \frac{s}{1+n_0} f(k_t) + \left(\frac{1-\delta}{1+n_0}\right) k_t \tag{27}$$

$$(B): k_{t+1} = \frac{s}{1+n_t} f(k_t) + \left(\frac{1-\delta}{1+n_t}\right) k_t \tag{28}$$

and

$$(C): k_{t+1} = sf(k_t) + (1-\delta)\, k_t \tag{29}$$

respectively, and with the same initial condition:

$$k_0^1 = k_0^2 = k_0^3 = k_0$$

If $\hat{k}_n$ and $\hat{k}$ are solutions of the equations (A) and (C), then:

1. $k_t^1 \leq k_t^2 \leq k_t^3 \ \forall \ t \in \mathbb{N}$
2. if $k_0 < \hat{k}_n$ then $\left(k_t^2\right)_{t\in\mathbb{N}}$ is strictly increasing in $[0, +\infty)$. In the case that $k_0 = \hat{k}_n$ then $\left(k_t^2\right)_{t\in\mathbb{N}}$ is increasing in $[0, +\infty)$
3. if $\hat{k}_n < k_0 \leq \hat{k}$ then $\exists \ \hat{t} \in \mathbb{N}$ such that: $\left(k_t^2\right)_{t\in\mathbb{N}}$ is decreasing $\forall t \leq \hat{t}$ and is strictly increasing $\forall t > \hat{t}$
4. if $\hat{k} < k_0$ then $\left(k_t^2\right)_{t\in\mathbb{N}}$ is strictly decreasing $\forall t \geq 0$ or $\exists \ \hat{t} \in \mathbb{N}$ such that: $\left(k_t^2\right)_{t\in\mathbb{N}}$ is strictly decreasing $\forall t \leq \hat{t}$ and is increasing $\forall t > \hat{t}$

Proof. 1. It is a consequence of $0 \leq n_t \leq n_0 \ \forall \ t \in \mathbb{N}$ and lemma 2.

2. It must be prove that $k_t^2 < k_{t+1}^2 \forall t \geq 0$, if $k_0 < \hat{k}_n$. By the definition of k_t^2, we have:

$$k_1^2 = \frac{s}{1+n_0} f(k_0^2) + \frac{(1-\delta)}{1+n_0} k_0^2 = \frac{s}{1+n_0} f(k_0^1) + \frac{(1-\delta)}{1+n_0} k_0^1 = k_1^1 \tag{30}$$

But since $k_0 < \hat{k}_n, k_1^1 > k_0^1 = k_0^2$, then: $k_1^2 > k_0^2$. For $t = 0$, the inequation is proved. Suppose now that it is true for $t = h-1$, it will be proved that $k_{h+1}^2 > k_h^2$:

$$k_{h+1}^2 = \frac{s}{1+n_h} f(k_h^2) + \frac{(1-\delta)}{1+n_h} k_h^2 > \frac{s}{1+n_{h-1}} f(k_{h-1}^2) + \frac{(1-\delta)}{1+n_{h-1}} k_{h-1}^2 = k_h^2 \tag{31}$$

because: $k_h^2 > k_{h-1}^2$ by induction hypothesis, f is monotonically increasing and $n_h \leq n_{h-1}$.

It must be studied now when $k_0 = \hat{k}_n$. Once again by (30) and since for this case we have $k_1^1 = k_0$, then $k_1^2 = k_0 = k_0^2$. If we continue:

$$k_2^2 = \frac{s}{1+n_1} f(k_1^2) + \frac{(1-\delta)}{1+n_1} k_1^2 \geq \frac{s}{1+n_0} f(k_0^2) + \frac{(1-\delta)}{1+n_0} k_0^2 = k_1^2 \tag{32}$$

Suppose now that $\exists\ h$ such that $k_h^2 \geq k_{h-1}^2$. Then with the same algebra of (31) it can be proved that $k_h^2 \geq k_{h-1}^2$ and then: $k_t^2 \leq k_{t+1}^2\ \forall t \geq 0$.

3. First observe that $\hat{k}_n < k_0$ and then:

$$k_1^2 = \frac{s}{1+n_0} f(k_0^2) + \frac{(1-\delta)}{1+n_0} k_0^2 = \frac{s}{1+n_0} f(k_0^1) + \frac{(1-\delta)}{1+n_0} k_0^1 = k_1^1 < k_0^1 = k_0^2 \tag{33}$$

Therefore $k_1^2 < k_0^2$. It must be proved that exits $\hat{t} \in \mathbb{N}$ such that $\left(k_t^2\right)_{t\in\mathbb{N}}$ is decreasing $\forall t \leq \hat{t}$ and is monotonically increasing $\forall t > \hat{t}$. Suppose that $k_{t+1}^2 \leq k_t^2\ \forall t \geq 0$. So we have:

$$k_{t+1}^2 \leq k_t^2\ \forall t \geq 0 \Leftrightarrow k_{t+1}^2 - k_t^2 \ \leq 0\ \forall t \geq 0 \Leftrightarrow \frac{s}{1+n_t} f(k_t^2) + \frac{(1-\delta)}{1+n_t} k_t^2 - k_t^2 \ \leq 0\ \forall t \geq 0 \Leftrightarrow \tag{34}$$

$$\Leftrightarrow s f(k_t^2) - (\delta + n_t) k_t^2 \leq 0\ \forall t \geq 0 \tag{35}$$

since $\left(k_t^2\right)_{t\in\mathbb{N}}$ is decreasing and bounded, then by the first part of this theorem have limit. Let $\bar{k} = \lim_t k_t^2$. Since $k_1^2 < k_0^2$ and $\left(k_t^2\right)_{t\in\mathbb{N}}$ is decreasing, we have that: $\hat{k}_n \leq \bar{k} < k_0 \leq \hat{k}$. Then:

$$\lim_t s f(k_t^2) - (\delta + n_t) k_t^2 = s f(\bar{k}) - \delta \bar{k} \leq 0 \Rightarrow \frac{f(\bar{k})}{\bar{k}} \leq \frac{\delta}{s} = \frac{f(\hat{k})}{\hat{k}} \tag{36}$$

Suppose now that $\bar{k} < \hat{k}$, so: $s f(\bar{k}) - \delta \bar{k} > 0$ and then $\dfrac{f(\bar{k})}{\bar{k}} > \dfrac{\delta}{s}$, that contradicts (36). Thus we can conclude that: $\bar{k} \geq \hat{k}$. But this inequality contradicts that $\hat{k}_n \leq \bar{k} < k_0 \leq \hat{k}$. Finally, we proved that $\exists\ t_0 > 0$ with $k_{t_0+1}^2 > k_{t_0}^2$. Let $\hat{t}$ the first natural that verify the last inequality. It must be proved now that $k_{t+1}^2 > k_t^2\ \forall t > \hat{t}$. For $t = \hat{t}$ that relation is true by definition. Suppose now that is true for $t = h-1 > \hat{t}$. Then:

$$k_{h+1}^2 = \frac{s}{1+n_h} f(k_h^2) + \frac{(1-\delta)}{1+n_h} k_h^2 > \frac{s}{1+n_{h-1}} f(k_{h-1}^2) + \frac{(1-\delta)}{1+n_{h-1}} k_{h-1}^2 = k_h^2$$

Therefore, $k_{t+1}^2 > k_t^2\ \forall t > \hat{t}$.

4. The proof is similar to the previous parts. ■

Theorem 4. If $(k_t)_{t\in\mathbb{N}}$ is a solution of $k_{t+1} = \frac{s}{1+n_t} f(k_t) + \left(\frac{1-\delta}{1+n_t}\right) k_t$, then:

$$\lim_{t\to+\infty} k_t = \hat{k}.$$

Proof. From the previous theorem we know that $(k_t)_{t\in\mathbb{N}}$ is monotone and bounded. This implies that $\lim_{t\to+\infty} k_t = \bar{k} < \infty$. Then we have that:

$$\bar{k} = \hat{k} \Leftrightarrow sf(\bar{k}) - \delta\bar{k} = 0.$$

From now on, we will suppose that $(k_t)_{t\in\mathbb{N}}$ is strictly decreasing for all $t \geq t_0$. (The proof is similar when $(k_t)_{t\in\mathbb{N}}$ is strictly increasing). Then we have that:

$$k_{t+1} \leq k_t \ \forall t > t_0$$

or equivalently:

$$sf(k_t) - (\delta + n_t)k_t \leq 0 \ \forall t > t_0.$$

Taking limits we have that:

$$sf(\bar{k}) - \delta\bar{k} \leq 0.$$

Suppose that $sf(\bar{k}) - \delta\bar{k} < 0$, and let $A = sf(\bar{k}) - \delta\bar{k} < 0$. Then $\exists t_1 > t_0$ such that:

$$sf(k_t) - (\delta + n_t)k_t < \frac{A}{2} \ \forall t \geq t_1$$

or equivalently

$$k_{t+1} - k_t < \frac{A}{2} \forall t \geq t_1$$

Then $\forall n > t_1$ we have that

$$\sum_{t=t_1}^{n} (k_{t+1} - k_t) < \sum_{t=t_1}^{n} \frac{A}{2} = \frac{A}{2} \frac{(n - t_1 + 1)(n + t_1)}{2}$$

implying that

$$k_{n+1} - k_{t_1} < \frac{A}{2} \frac{(n - t_1 + 1)(n + t_1)}{2}, \forall n > t_1.$$

Taking limits we obtain:

$$\lim_{n} (k_{n+1} - k_{t_1}) \leq \lim_{n} \frac{A}{2} \frac{(n - t_1 + 1)(n + t_1)}{2} = -\infty,$$

This contradicts the fact that k_t is bounded. Then we have that $sf(\bar{k}) - \delta\bar{k} = 0$. ■

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