E C O N O M I C S B U L L E T I N

The Generalized Gini index and the measurement of income mobility

Jacques SILBER
Bar-Ilan University

Michal Weber
Bar-Ilan University

Abstract

Two new normative indices of mobility are proposed. The first one is a population weighted generalized Gini mobility index and will be higher, the higher the size of the transfer between two individuals and, for a given transfer, the higher the rank difference between the individuals between whom the transfer takes place. This index is also higher, the greater the rank gap between the individuals between whom a swap takes place. The second index is an income weighted generalized Gini mobility index. When a transfer takes place between two individuals this index will be higher, the greater the transfer. Similarly in the case of a swap between the incomes of two individuals, the index will be higher, the greater the gap between the incomes of the two individuals between whom the incomes are swapped. The empirical illustration is based on Israeli Census data.

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1. Introduction

Using Atkinson's (1970) concept of "equally distributed equivalent level of income" and following earlier work by Blackorby and Donaldson (1978), Donaldson and Weymark (1980) and Yitzhaki (1983) independently developed an extension of the Gini index that allowed in a certain way to specify the weight one wanted to give to low incomes when measuring inequality.
Another extension of the Gini index was proposed by Silber (1995) when he showed how the Gini index could be the basis for measuring distributional change. Silber (1995) proposed in fact two new indices of distributional change, a population- and an income-weighted index, that were derived from the Gini index and whose properties were quite similar to those of the entropy related index of distributional change proposed by Cowell (1985). These indices of distributional change may in fact be considered as indices measuring the degree of income mobility (see, Fields and Ok, 1999, for a thorough review of measures of income mobility).
The purpose of this note is to combine Donaldson and Weymark's (1980) approach to extending Gini's inequality index with Silber's (1995) application of the Gini index to the measurement of distributional change or income mobility. As in Silber (1995) a population- and an income-weighted mobility index is proposed. This generalization of Gini related mobility indices will allow us to decide either how much weight to give to individuals who were originally poor (for the population weighted mobility indices) or how much weight to give to those individuals who experienced a low rate of growth of income (for the income weighted mobility indices).
In the following sections population- and income weighted generalized Gini mobility indices are defined. Then an empirical illustration based on Israeli Census data for the years 1983 and 1995 is presented.

2. Population weighted Generalized Gini Mobility Indices

Recall that the Generalized Gini index may be expressed (see Donaldson and Weymark, 1980) as

$$I_{GG} = 1 - [\sum_{i=1}^{n} \frac{(i^{\delta} - (i-1)^{\delta})}{n^{\delta}} \frac{s_i}{f_i}] \qquad (1)$$

where $f_i = (1/n)$ represents the share in the total population (n) of the individual earning income y_i, $s_i = (y_i / n\bar{y})$ refers to the share in total income of the individual earning income y_i, $\bar{y}$ is the average income and it is assumed that $y_1 \geq \dots y_i \geq \dots y_n$.
Assume now that instead of comparing "prior" population shares f_i with "posterior" income shares s_i, we compare income shares s_i at time 0 with income shares w_i at time 1. We continue to assume that the sub index i refers to only one individual, $\forall i$.
We will also suppose that the individuals are ranked by decreasing values of the ratios (s_i / f_i), that is by decreasing values of s_i and hence of y_i since it is assumed that $f_i = (1/n)$ $\forall i$. Note that the shares w_i at time 1 will be ranked like the shares s_i the individuals had at time 0.
Generalizing the approach to mobility measurement suggested by Silber (1995) we can now define a population weighted Gini mobility index J_{GGP} as

$J_{GGP} = \{1-[\sum_{i=1}^{n}\frac{((i^{\delta}-(i-1)^{\delta})}{n^{\delta}}\frac{s_i}{f_i}]\}-\{1-[\sum_{i=1}^{n}\frac{((i^{\delta}-(i-1)^{\delta})}{n^{\delta}}\frac{w_i}{f_i}]\} =$ $[\sum_{i=1}^{n}\frac{((i^{\delta}-(i-1)^{\delta})}{n^{\delta}}\frac{w_i}{f_i}]-[\sum_{i=1}^{n}\frac{((i^{\delta}-(i-1)^{\delta})}{n^{\delta}}\frac{s_i}{f_i}] =$ $\sum_{i=1}^{n}\frac{((i^{\delta}-(i-1)^{\delta})}{n^{\delta}}\frac{(w_i-s_i)}{f_i}$	(2)

It is easy to show that when $\delta = 2$, expression (2) gives the population weighted Gini index of mobility defined by Silber (1995).

The mobility index J_{GGP} has the desirable properties of a mobility index. First, if $w_i = s_i \quad \forall i, \ J_{GGP} = 0$. Second, we prove in Appendix A that when a given sum is transferred from a richer to a poorer individual, the mobility index J_{GGP} will be positive and higher, the greater the amount transferred and the rank difference between the individuals between whom the transfer takes place. Similarly when there is a swap between the incomes of two individuals, Appendix A shows that J_{GGP} will be higher, the greater the rank difference between the individuals between whom the swap takes place.

3. Income weighted Generalized Gini Mobility Indices

Using notations previously given let us now define the "prior" shares as the income shares s_i at time 0 while the "posterior" shares will be the income shares w_i at time 1. We continue to assume that the subindex i refers to only one individual, $\forall i$.

This time however we are going to rank the individuals by decreasing ratios $(\frac{w_i}{s_i})$.

Using (1) the income weighted generalized mobility index J_{GGI} will then be expressed as

$J_{GGI} = 1-[\sum_{i=1}^{n}((\sum_{j=1}^{i}s_j)^{\delta}-(\sum_{j=1}^{i-1}s_j)^{\delta})]\frac{w_i}{s_i}$	(3)

Assume, without loss of generality, that originally the individuals were ranked by decreasing incomes so that $s_1 \geq \ldots s_i \geq \ldots s_n$.

In Appendix B we show that if a given sum is transferred from a richer to a poorer individual, the income weighted mobility index J_{GGI} will be higher, the greater the amount transferred. Similarly it can be proven that when the incomes of two individuals are swapped, the greater the income gap between the individuals whose incomes are swapped, the greater the value of the income weighted mobility index J_{GGI}.

4. An Empirical Illustration

The empirical illustration that will be presented here is based on the Israeli Censuses for the years 1983 and 1995.

Table 1 gives the value of the population weighted generalized Gini mobility index for three ethnic origins (the database we used included only Jews so that the "ethnic origin" refers in fact to the continent of birth). We divided the population in three groups: those born in Asia or Africa (Easterners), in Israel and in Europe or America (Westerners). For the parameter δ we chose $\delta = 2$, $\delta = 3$ and $\delta = 10$. We know that the greater δ (remember that $\delta \geq 2$), the greater the weight given to mobility among poorer individuals. Table 1 gives also confidence intervals based on the bootstrap technique for this population weighted generalized Gini mobility index. It appears that there are no significant differences between the three ethnic origins when $\delta = 2$ and $\delta = 3$. The index is however significantly lower among Westerners when $\delta = 10$.

Table 1: Population weighted generalized Gini mobility indices

	Easterners	Individuals born in Israel	Westerners
Number of observations	2003	3226	1677
Value of parameter δ	**The value of the population weighted mobility Index (Bootstrap confidence interval for indicator in parenthesis)**		
$\delta = 2$	0.2 (0.17-0.22)	0.21 (0.19-0.23)	0.19 (0.17-0.22)
$\delta = 3$	0.25 (0.21-0.28)	0.25 (0.23-0.28)	0.23 (0.2-0.26)
$\delta = 10$	0.35 (0.3-0.4)	0.36 (0.31-0.41)	0.29 (0.24-0.34)

Table 2 gives the value of the income weighted generalized mobility index for the three ethnic origins when $\delta = 2$, $\delta = 3$ and $\delta = 10$. It appears that income mobility is significantly lower among Easterners than among individuals born in Israel when $\delta = 2$ and $\delta = 3$. When $\delta = 10$ there is no significant difference between the three groups.

Table 2: Income weighted generalized Gini mobility indices

	Easterners	Individuals born in Israel	Westerners
Number of observations	2003	3226	1677
Value of parameter δ	**The value of the income weighted generalized mobility Index (Bootstrap confidence interval for indicator in parenthesis)**		
$\delta = 2$	0.39 (0.38-0.42)	0.42 (0.4-0.44)	0.41 (0.39-0.43)
$\delta = 3$	0.52 (0.51-0.56)	0.56 (0.53-0.58)	0.55 (0.52-0.57)
$\delta = 10$	0.78 (0.77-0.83)	0.81 (0.78-0.83)	0.82 (0.79-0.84)

5. Conclusion

In this study we introduced two new normative indices of mobility. We proved that the population weighted generalized Gini mobility index will be higher, the higher the size of the transfer between two individuals and, for a given transfer, the higher the rank difference between the individuals between whom the transfer takes place. In the case of a swap we proved that this population weighted mobility index will be higher, the greater the rank gap between the individuals between whom the swap takes place. In the case of the income weighted generalized Gini mobility index we proved that when a transfer takes place between two individuals the index will be higher, the greater the transfer. Similarly in the case of a swap between the incomes of two individuals, the index will be higher, the greater the gap between the incomes of the two individuals between whom the incomes are swapped. The empirical illustration showed that the choice of the parameter δ had an impact on differences in mobility between ethnic groups.

References

Atkinson, A.B. (1970) "On the Measurement of Inequality" *Journal of Economic Theory* **2**: 244-263**.**

Blackorby, C. and D. Donaldson (1978) "Measures of Relative Equality and Their Meaning in Terms of Social Welfare" *Journal of Economic Theory* **18**:59-80.

Cowell, F.A. (1985) "Measures of Distributional Change: An Axiomatic Approach" *Review of Economic Studies* **52**: 135-151.

Donaldson, D. and J. A. Weymark (1980) "A Single Parameter Generalization of the Gini Indices of Inequality" *Journal of Economic Theory* **22**: 67-88.

Fields, G.S. and E.A. Ok (1999) "The Measurement of Income Mobility: An Introduction to the Literature" in *Handbook on Income Inequality Measurement* by J. Silber, Ed., Kluwer Academic Publishers, Dordrecht: 557-596.

Silber, J. (1995) "Horizontal Inequity, the Gini Index, and the Measurement of Distributional Change" in *Income Distribution, Social Welfare, Inequality and Poverty, Research on Economic Inequality* by C. Dagum and A Lemmi, Eds., **6**: 379-392.

Yitzhaki, S. (1983) "On an Extension of the Gini Inequality Index" *International Economic Review* **24** (3): 617-628.

Appendix A: Properties of the population weighted generalized Gini mobility indices

The case of a transfer a from a richer individual h to a poorer individual k.

Assume that $w_i = s_i \quad \forall i \neq h,k$, $w_h = s_h - a$ and $w_k = s_k + a$

We then may write that $w_k - s_k = a$ and that $w_h - s_h = -a$

The population generalized weighted Gini mobility index J_{GGP} may then be expressed as

$J_{GGP} = [\frac{(h^\delta - (h-1)^\delta)}{n^\delta}(-a)] + [\frac{(k^\delta - (k-1)^\delta}{n^\delta}(+a)] =$ $a[\frac{(k^\delta - (k-1)^\delta)}{n^\delta} - \frac{(h^\delta - (h-1)^\delta)}{n^\delta}]$	(A-1)

Call β_h the expression $\frac{(h^\delta - (h-1)^\delta)}{n^\delta}$ and β_k the expression $\frac{(k^\delta - (k-1)^\delta}{n^\delta}$.

Expression (A-1) may then be expressed as

$J_{GGP} = a(\beta_k - \beta_h)$	(A-2)

with clearly $\beta_h \leq \beta_k$ since $k > h$ and $\delta > 2$, so that J_{GGP} is positive and will be higher, the higher the transfer a. Note also that J_{GGP} will be higher, the greater the rank difference between the individuals h and k between whom the transfer takes place, since $(\beta_k - \beta_h)$ increases with this rank difference.

The Case of a Swap

Assume now that $w_i = s_i \quad \forall i \neq h,k$, $w_h = s_k$ and $w_k = s_h$.

The population weighted Gini mobility index J_{GGP} may then be expressed as

$J_{GGP} = [\frac{(h^\delta - (h-1)^\delta)}{n^\delta}(s_k - s_h) + \frac{(k^\delta - (k-1)^\delta)}{n^\delta}(s_h - s_k)$	(A-3)

Call now b the gap $(s_h - s_k)$.

Expression (A-3) may then be expressed as $J_{GGP} = b(\beta_k - \beta_h)$ which is positive and higher, the higher b. Note also that J_{GGP} will be higher, the greater the rank difference between the individuals h and k between whom the swap takes place, since $(\beta_k - \beta_h)$ increases with this rank difference.

Appendix B: Properties of the income weighted generalized Gini mobility indices.

The case of a transfer a from a richer individual h to a poorer individual k.

Assume that $w_i = s_i \quad \forall i \neq h,k$, $w_h = s_h - a$ and $w_k = s_k + a$

We can therefore conclude that, once the transfer has taken place, $(\frac{w_k}{s_k}) > 1 > (\frac{w_h}{s_h})$.

For all $i \neq h,k$ we may write that $(\frac{w_i}{s_i}) = 1$. Given that in the case of an income weighted mobility index we have to order the individuals by decreasing ratios $(\frac{w_i}{s_i})$, the order of the individuals will now be: $k, 1, 2, \dots, (h-1), (h+1), \dots, n, h$.

Expression (3) may therefore be written as

$$J_{GGI} = 1 - \{[(s_k)^\delta - 0](\frac{w_k}{s_k}) + \sum_{i \neq h,k}[(\sum_{j=1}^{i} s_j)^\delta - (\sum_{j=1}^{i-1} s_j)^\delta](1) + [(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n + s_h)^\delta - (s_k + s_1 + \dots + s_{h-1} + s_{h+1} \dots + s_n)^\delta](\frac{w_h}{s_h})\} \quad \text{(B-1)}$$

$$J_{GGI} = 1 - \{[(s_k)^\delta - 0](1 + \frac{a}{s_k}) + \sum_{i \neq h,k}[(\sum_{j=1}^{i} s_j)^\delta - (\sum_{j=1}^{i-1} s_j)^\delta](1) + [(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n + s_h)^\delta - (s_k + s_1 + \dots + s_{h-1} + s_{h+1} \dots + s_n)^\delta](1 - \frac{a}{s_h})\} \quad \text{(B-2)}$$

It is easy to observe that after simplifying we end up with

$$\leftrightarrow J_{GGI} = [-(s_k)^\delta(\frac{a}{s_k})] + [(1 - (s_k + s_1 + \dots + s_{h-1} + s_{h+1} \dots + s_n)^\delta)(\frac{a}{s_h})$$
$$\leftrightarrow J_{GGI} = a\{[\frac{1 - (s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)^\delta}{s_h}] - [\frac{(s_k)^\delta}{s_k}]\} \quad \text{(B-3)}$$

Now, given that $(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n) < 1$, we derive that
$(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)^\delta < (s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)$ so that
$1 - (s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)^\delta > 1 - (s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)$.

But since $1-(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n) = s_h$ we can now conclude that $\frac{1-(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)^\delta}{s_h} > \frac{s_h}{s_h} = 1$. On the other hand we also know that $\frac{(s_k)^\delta}{s_k} = (s_k)^{\delta-1} < 1$ since $s_k < 1$.

We can therefore conclude that

$\frac{1-(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)^\delta}{s_h} > 1 > (s_k)^{\delta-1}$	(B-4)

so that the expression with which a is multiplied on the R.H.S of (B-3) is greater than one and, as a consequence, the greater the transfer a, the greater the value of the income weighted mobility index J_{GGI}.

The case of a Swap

Assume as before that originally $s_1 \geq \dots s_i \geq \dots s_n$. Now we also assume that $w_i = s_i \quad \forall i \neq h,k$, $w_h = s_k$ and $w_k = s_h$

Since we assumed that $s_h > s_k$, and remembering that in (3) the ratios $(\frac{w_i}{s_i})$ are ranked by decreasing values, following the swap, the ranking of the individuals will be, here again: $k,1,2,\dots(h-1),(h+1),\dots n,h.$

so that expression (3) will now be written as

$J = 1-\{[((s_k)^\delta - 0)(\frac{s_h}{s_k})] + [\sum_{i \neq h,k}[(\sum_{j=1}^{i} s_j)^\delta - (\sum_{j=1}^{i-1} s_j)^\delta](1)]$ $+[[(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n + s_h)^\delta - (s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)^\delta](\frac{s_k}{s_h})]\}$	(B-5)

Assume now that $s_h - s_k = \lambda$. We therefore derive that $(\frac{s_h}{s_k}) = (1+\frac{\lambda}{s_k})$ and that $(\frac{s_k}{s_h}) = (1-\frac{\lambda}{s_h})$.

Expression (3) will now be written as

$J_{GGI} = 1-\{[[(s_k)^\delta - 0](1+(\frac{\lambda}{s_k}))] + [\sum_{i \neq h,k}[(\sum_{j=1}^{i} s_j)^\delta - (\sum_{j=1}^{i-1} s_j)^\delta](1)]$ $+[[(s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n + s_h)^\delta - (s_k + s_1 + \dots s_{h-1} + s_{h+1} \dots + s_n)^\delta](1-(\frac{\lambda}{s_h}))]\}$	(B-6)

Since expression (B-2) and (B-6) are very similar, we will not repeat the demonstration and we can therefore conclude that the greater the income gap between the individuals whose incomes are swapped (individuals h and k), the greater the value of the income weighted mobility index J_{GGI}.