Economics Bulletin

Volume 29, Issue 3

Characterizing the Nash social welfare relation for infinite utility streams: a note

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Abstract

This note provides an axiomatic analysis of a social welfare ordering over infinite utility streams. We offer two characterizations of an infinite-horizon version of the Nash criterion.

I am extremely grateful to an associate editor and an anonymous referee of the journal for valuable comments. Needless to say, I am responsible for any remaining errors. Further, I gratefully acknowledge the Japan Society for the Promotion of Science (JSPS) Research Fellowships for Young Scientists and Grant-in-Aids for JSPS Fellows of the Ministry of Education, Culture, Sports, Science and Technology, Government of Japan.
Citation: Susumu Cato, (2009) "Characterizing the Nash social welfare relation for infinite utility streams: a note", *Economics Bulletin*, Vol. 29 no.3 pp. 2372-2379.
Submitted: Mar 26 2008. **Published:** September 18, 2009.

1 Introduction

Ranking infinite utility streams has become one of the major topics in social choice theory. The fundamental impossibility result is given by Diamond (1965). He shows that a social welfare ordering satisfying Pareto and anonymity cannot be continuous in the topology induced by the supremum norm. Similar difficulties are obtained by Campbell (1985), Lauwers (1997a), Shinotsuka (1997), Basu and Mitra (2003), Fleurbaey and Michel (2003), Sakai (2006), and Hara et al. (2008).

Recent studies indicate that if we dispense with continuity assumption, then we can avoid impossibility results, and obtain various characterization results: Basu and Mitra (2007) propose and characterize the *utilitarian* social welfare relation for infinite utility streams; Bossert et al. (2007) give definitions and characterizations of the *generalized Lorenz* and *leximin* social welfare relations for infinite utility streams.

In this note, we propose an infinite-horizon version of the *Nash* criterion and provide two characterizations of such a relation. Our first characterization is obtained by an ethical principle[1]: a social welfare ordering satisfies *Pareto* and *ratio-incremental equity* if and only if it is an ordering extension of the Nash social welfare relation (Theorem 1). Our second characterization is obtained by an invariance axiom: a social welfare ordering satisfies *Pareto*, *anonymity*, and *partial ratio-scale invariance* if and only if it is an ordering extension of the Nash social welfare relation (Theorem 2). In the proof of Theorem 2, we apply the result of Basu and Mitra (2007).

This note is organized as follows: Section 2 presents our notation and definitions. Section 3 proposes a Nash social welfare relation for infinite utility streams. Section 4 presents our results. Section 5 concludes the paper. All proofs are relegated to the Appendix.

2 Notation and Definitions

Let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, 3, \cdots\}$. Let $\mathbb{R}_{++}$ be the set of all positive real numbers. The set of infinite utility streams is $X = \mathbb{R}_{++}^{\mathbb{N}}$.[2] We write $x = (x_1, x_2, x_3, \dots)$ to denote an element of X. Given an infinite utility stream $x \in X$, for $n \in \mathbb{N}$, we define the *n-head* of x by

$$x^{-n} = (x_1, \dots, x_n)$$

and we define the *n-tail* of x by

$$x^{+n} = (x_{n+1}, x_{n+2}, \dots).$$

[1] Fleurbaey and Michel (2001), and Sakai (2003a) propose interesting ethical principles and investigate their implications. See also Sakai (2003b).

[2] This assumption is essential for our analysis. If the set of infinite utility streams is $X = \mathbb{R}$ or $X = \mathbb{R}_+$, difficulties arise. One of the most important points is that a Nash social welfare function does not satisfy the Pareto axiom for such cases. For example, consider the following the two streams: $x = (0, 0, 1, 1, 1, \dots)$ and $y = (1, 0, 1, 1, 1, \dots)$. By our definition of a Nash social welfare function, the two streams are indifferent. However, the (strong) Pareto axiom implies that y is strictly preferred to x.

We must explain why the set of utilities is restricted in this way. One possible interpretation is that x represents the difference with a minimum necessary for human survival. For example, each generation dies when $u \leq \underline{u}$, and the government must guarantee utilities that are strictly larger than $\underline{u}$. Thus, we redefine the set of utilities: $x := u - \underline{u} > 0$.

Moreover, the requirement of positive utility introduces a degree of cardinality. This fact leads us to another interpretation of x_i: each value x_i represents the income level of generation i. In this case, $\underline{u}$ is the critical income level.

For all $x, y \in X$, $x + y = (x_1 + y_1, x_2 + y_2, \dots)$. For all $x, y \in X$, $x \cdot y = (x_1 y_1, x_2 y_2, \dots)$. A constant sequence satisfies $x_i = a$ for all $i \in \mathbb{N}$ for some $a \in \mathbb{R}_+$, and it written as $(a)_{con}$

For $x, y \in X$, $x \geq y$ if $x_i \geq y_i$ for all $i \in \mathbb{N}$. For $x, y \in X$, $x > y$ if $x \geq y$ and $x \neq y$.

A *social welfare relation* is a binary relation $\succeq$ on X, which is reflexive and transitive. The symmetric and asymmetric part of $\succeq$ is defined as usual sense. Hence, $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$; and, $x \succ y$ if and only if $x \succeq y$ and $\neg(y \succeq x)$. A *social welfare ordering* is a binary relation $\succeq$ on X, which is reflexive, complete and transitive. Let $\succeq_S$ and $\succeq_T$ be social welfare relations. If $\succeq_S \Rightarrow \succeq_T$ and $\succ_S \Rightarrow \succ_T$, we call $\succeq_T$ is an *extension* of $\succeq_S$. If an extension $\succeq_T$ of $\succeq_S$ is an ordering, we call $\succeq_T$ an *ordering extension* of $\succeq_S$.

A *finite permutation* π is a permutation, such that there exists $m \in \mathbb{N}$ with $\pi(i) = i$ for all $i \geq m$. We write $\pi(x)$ for the vector $(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(i)}, \dots)$.

3 The Nash Social Welfare Relation

We propose an infinite-horizon version of the Nash criterion. Let us define a social welfare relation $\succeq_N$ on X by

$$x \succeq_N y \text{ if and only if } \exists n \in \mathbb{N} \text{ such that } (\prod_{i=1}^{n} x_i, x^{+n}) \geq (\prod_{i=1}^{n} y_i, y^{+n}).$$

Note that a binary relation $\succeq_N$ is reflexive and transitive, but it is not necessarily complete. This definition is a simple extension of the standard definition of the Nash criterion. For the argument for the finite version of the Nash criterion, see Kaneko and Nakamura (1979) and Roberts (1980).

A social welfare relation $\succeq_N$ has the following properties:

$$\exists n \in \mathbb{N} \text{ such that } (\prod_{i=1}^{n} x_i, x^{+n}) \geq (\prod_{i=1}^{n} y_i, y^{+n}) \Rightarrow \forall m > n, (\prod_{i=1}^{m} x_i, x^{+m}) \geq (\prod_{i=1}^{m} y_i, y^{+m}),$$
$$\exists n \in \mathbb{N} \text{ such that } (\prod_{i=1}^{n} x_i, x^{+n}) > (\prod_{i=1}^{n} y_i, y^{+n}) \Rightarrow \forall m > n, (\prod_{i=1}^{m} x_i, x^{+m}) > (\prod_{i=1}^{m} y_i, y^{+m}).$$

By the contributions of Arrow (1951) and Szpilrajn (1930), we know that every binary relation that is reflexive and transitive has an ordering extension. Therefore, there exists a social welfare ordering $\succeq$ that is an ordering extension of $\succeq_N$.

4 The Results

We introduce four axioms on $\succeq$. The following axiom is well-known and therefore requires no explanation.

Pareto: For all $x, y \in X$, $x > y \Rightarrow x \succ y$.

Next, we propose ratio-incremental equity.

Ratio-incremental equity: For all $x, y \in X$, for all $s, t \in \mathbb{N}$, and for all $\epsilon \in \mathbb{R}_{++}$, if (i) $[y_s = x_s \epsilon \wedge y_t = x_t / \epsilon]$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{s, t\}$, then $x \sim y$.

This is an equity axiom that requires an impartial treatment of a utility ratio change.[3] This axiom has a role similar to the incremental equity axiom proposed by Blackorby et al. (2002).[4]

The following axiom requires the equal treatment of generations.

Anonymity: For all $x \in X$ and all finite permutations π of $\mathbb{N}$, $x \sim \pi(x)$.

Note that our definition of anonymity does not allow an infinite permutation. Lauwers (1997b, 1997c) and Mitra and Basu (2007) discuss classes of permutations that include infinite permutations.[5]

The following axiom is an adaptation of the invariance transformation condition used in classical social choice theory.[6] This axiom is an appropriate counterpart of partial-unit comparison, introduced by Basu and Mitra (2007).

Partial ratio-scale invariance: For all $x, y, a \in X$, and for all $n \in \mathbb{N}$, if $x^{+n} = y^{+n}$ and $x \succeq y$, then $x \cdot a \succeq y \cdot a$.

We present our results. Our first result characterizes all ordering extensions of $\succeq_N$ by ratio-incremental equity.

Theorem 1. A social welfare ordering $\succeq$ on X satisfies Pareto and ratio-incremental equity if and only if $\succeq$ is an ordering extension of $\succeq_N$.

Note that in this characterization, we do not impose anonymity.

Our second result characterizes all ordering extensions of $\succeq_N$ by partial ratio-scale invariance.

Theorem 2. A social welfare ordering $\succeq$ on X satisfies Pareto, anonymity, and partial ratio-scale invariance if and only if $\succeq$ is an ordering extension of $\succeq_N$.

5 Concluding Remarks

In this note, we characterize the Nash social welfare relation for infinite utility streams in two ways. In our first characterization, the key axiom is ratio-incremental equity. This axiom is in the spirit of impartiality assumptions emphasized by many authors. Bossert et al. (2007) investigate two classical equity axioms, which have an ethical motivation. They characterize the infinite version of the generalized Lorenz criterion and of leximin by the Pigou-Dalton equity principle and the Hammond equity principle, respectively. In our second characterization, the key axiom is partial ratio-scale invariance. This axiom specifies the informational structure of individual's utilities.

[3]An anonymous referee points out the problem of the ethical appeal of ratio-incremental equity as the impartiality of utility ratio changes. Consider $x = (a)_{con} \in X$ for some positive constant $a \in \mathbb{R}_{++}$. Let ϵ be a very large positive value. Let $y \in X$ be such that $[y_s = x_s\epsilon \wedge y_t = x_t/\epsilon]$ and $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{s, t\}$. The ratio-incremental equity requires that $x \sim y$ for any ϵ. Note that $y_s \to \infty$ and $y_t \to 0$ as $\epsilon \to \infty$. These facts imply that ratio-incremental equity allows inequality among individuals.

[4]Kamaga and Kojima (2008a) characterize an infinite version of utilitarian social welfare relation by the incremental equity axiom proposed by Blackorby et al. (2002). Their characterization is closely related to Theorem 1 above. They also investigate the extended anonymity axiom, which is proposed and studied in Mitra and Basu (2007), and characterize the extended versions of the generalized Lorenz and leximin criteria.

[5]See also Banerjee (2006), Kamaga and Kojima (2008a), and Kamaga and Kojima (2008b).

[6]For example, see Roberts (1980).

Appendix

Proof of Theorem 1. Sufficiency: Suppose that $\succeq$ is an ordering extension of $\succeq_N$. We check that Pareto is satisfied. Suppose that $x > y$. Obviously, we have that there exists $n \in \mathbb{N}$ such that $(\prod_{i=1}^n x_i, x^{+n}) > (\prod_{i=1}^n y_i, y^{+n})$. Then, $x \succ_N y$. Since $\succeq$ is an ordering extension of $\succeq_N$, we have $x \succ y$. Now, we show that ratio-incremental equity is satisfied. We consider two sequences $x, y \in X$ such that (i) $y_s = x_s\epsilon$ and $y_t = x_t/\epsilon$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{s, t\}$. Obviously, we have that for $n = \max\{s, t\} \in \mathbb{N}$, $(\prod_{i=1}^n x_i, x^{+n}) = (\prod_{i=1}^n y_i, y^{+n})$. Then, by definition of $\succeq_N$, $x \sim_N y$. Since $\succeq$ is an ordering extension of $\succeq_N$, we have $x \sim y$.

Necessity: Suppose that a social welfare ordering $\succeq$ satisfies Pareto and ratio-incremental equity. To prove $\succeq$ is an ordering extension of $\succeq_N$, we have to show that $\succeq_N \Rightarrow \succeq$ and $\succ_N \Rightarrow \succ$. We take $x, y \in X$ such that $x \succ_N y$. By definition of $\succeq_N$, there exists $n \in N$ such that

$$\prod_{i=1}^n x_i > \prod_{i=1}^n y_i \text{ and } x^{+n} \geq y^{+n}.$$

Now we prove that $x \succ y$. Ratio-incremental equity implies the following results:

$$\begin{aligned}
(x^{-n}, x^{+n}) &\sim \Big((\prod_{i=1}^n x_i)^{1/n}, x_2, \ldots, \frac{x_1 x_n}{(\prod_{i=1}^n x_i)^{1/n}}, x^{+n}\Big) \\
&\sim \Big((\prod_{i=1}^n x_i)^{1/n}, (\prod_{i=1}^n x_i)^{1/n}, \ldots, \frac{x_1 x_2 x_n}{(\prod_{i=1}^n x_i)^{2/n}}, x^{+n}\Big) \\
&\vdots \\
&\sim \Big((\prod_{i=1}^n x_i)^{1/n}, (\prod_{i=1}^n x_i)^{1/n}, \ldots, (\prod_{i=1}^n x_i)^{1/n}, x^{+n}\Big).
\end{aligned}$$

Therefore,

$$(x^{-n}, x^{+n}) \sim (\hat{x}^{-n}, x^{+n}) \tag{1}$$

where $\hat{x} = ((\prod_{i=1}^n x_i)^{1/n})_{con}$. By the same argument, we obtain

$$(y^{-n}, y^{+n}) \sim (\hat{y}^{-n}, y^{+n}) \tag{2}$$

where $\hat{y} = ((\prod_{i=1}^n y_i)^{1/n})_{con}$. Note that in this case, we have $(\prod_{i=1}^n x_i)^{1/n} > (\prod_{i=1}^n y_i)^{1/n}$ Hence, Pareto implies that $(\hat{x}^{-n}, x^{+n}) \succ (\hat{y}^{-n}, y^{+n})$. Therefore, in combination with (1) and (2), the transitivity of $\succeq$ implies that $x \succ y$.

Next, we take $x, y \in X$ such that $x \succeq_N y$. By definition of $\succeq_N$, there exists $n \in N$ such that

$$(\prod_{i=1}^n x_i, x^{+n}) \geq (\prod_{i=1}^n y_i, y^{+n}).$$

If $(\prod_{i=1}^n x_i, x^{+n}) > (\prod_{i=1}^n y_i, y^{+n})$, then $x \succ y$ by the above argument. Hence, we have to consider the case where $(\prod_{i=1}^n x_i, x^{+n}) = (\prod_{i=1}^n y_i, y^{+n})$. In this case, $(\hat{x}^{-n}, x^{+n}) = (\hat{y}^{-n}, y^{+n})$. Hence, by (1) and (2), the transitivity of $\succeq$ implies that $x \sim y$. Therefore, $\succeq$ is an ordering extension of $\succeq_N$. ■

Before proving Theorem 2, we refer to the result of Basu and Mitra (2007). They propose the following criterion. Let us define a social welfare relation $\succeq_U$ on $\mathbb{R}^{\mathbb{N}}$ by

$$x \succeq_U y \text{ if and only if } \exists n \in \mathbb{N} \text{ such that } (\sum_{i=1}^{n} x_i, x^{+n}) \geq (\sum_{i=1}^{n} y_i, y^{+n}).$$

Basu and Mitra (2007) introduce the following axiom.

Partial-unit comparability: For all $x, y, b \in \mathbb{R}^{\mathbb{N}}$, and for all $n \in \mathbb{N}$, if $x^{+n} = y^{+n}$ and $x \succeq y$, then $x + b \succeq y + b$.

They show that a social welfare ordering $\succeq$ on $\mathbb{R}^{\mathbb{N}}$ satisfies Pareto, anonymity, and partial-unit comparability if and only if it is an ordering extension of $\succeq_U$ (Basu and Mitra (2007), Theorem 1).[7]

Proof of Theorem 2. Sufficiency: Suppose that $\succeq$ is an ordering extension of $\succeq_N$. In the proof of Theorem 1, we have already checked that Pareto is satisfied. Now, we show that anonymity is satisfied. Let $x \in X$ and π be a finite permutation of $\mathbb{N}$. There exists $m \in \mathbb{N}$ such that $x_i = \pi(x_i)$ for all $i \geq m$. Obviously, we have that $(\prod_{i=1}^{m} x_i, x^{+m}) = (\prod_{i=1}^{m} y_i, y^{+m})$. Then, $x \sim_N y$. Since $\succeq$ is an ordering extension of $\succeq_N$, we have $x \sim y$. Finally, we show that ratio-scale invariance is satisfied. We take $x, y \in X$ such that $x^{+n} = y^{+n}$ for some $n \in \mathbb{N}$, and $x \succeq y$. Cleary, $\prod_{i=1}^{n} x_i \geq \prod_{i=1}^{n} y_i$. This implies that for all $a \in X$, $\prod_{i=1}^{n} x_i a_i \geq \prod_{i=1}^{n} y_i a_i$ and $x^{+n} \cdot a^{+n} = y^{+n} \cdot a^{+n}$. Since $\succeq$ is an ordering extension of $\succeq_N$, we have $x \cdot a \succeq y \cdot a$.

Necessity: Suppose that a social welfare ordering $\succeq^*$ on $\mathbb{R}^{\mathbb{N}}$ satisfies Pareto, anonymity, and partial-unit comparison. By Theorem 1 of Basu and Mitra (2007), if $\succeq^*$ satisfies Pareto, anonymity, and partial-unit comparability, then $\succeq^*$ is an ordering extension of $\succeq_U$. Now we define an ordering $\succeq$ on X as follows: for all $x, y \in X$,

$$(e^{x_1}, e^{x_2}, \dots) \succeq (e^{y_1}, e^{y_2}, \dots) \Leftrightarrow x \succeq^* y.$$

It is straightforward to show that $\succeq$ satisfies Pareto and anonymity. Furthermore, by taking $a_i = e^{b_i}$, we can check that $\succeq$ also satisfies ratio-scale invariance. By definition of $\succeq$, $x \succeq y$ holds if and only if $(\log x_1, \log x_2, \dots) \succeq^* (\log y_1, \log y_2, \dots)$. Since $\succeq^*$ is an ordering extension of $\succeq_U$, if there exists $n \in \mathbb{N}$ such that $(\sum_{i=1}^{n} \log x_i, \log x_{n+1}, \dots) \geq (\sum_{i=1}^{n} \log y_i, \log y_{n+1}, \dots)$, then $x \succeq y$. Note that $\sum_{i=1}^{n} \log x_i = \log \prod_{i=1}^{n} x_i$. This implies that $x \succeq_N y \Rightarrow x \succeq y$. Similarly, we can show that $x \succ_N y \Rightarrow x \succ y$. Therefore, $\succeq$ is an ordering extension of $\succeq_N$. ■

References

[1] Arrow, K.J. (1951) *Social Choice and Individual Values* Wiley, New York (2nd ed., 1963).

[2] Banerjee, K. (2006) "On the extension of the utilitarian and Suppes-Sen social welfare relations to infinite utility streams" *Social Choice and Welfare* **27**, 327–339.

[3] Basu, K., and T. Mitra (2003) "Aggregating infinite utility streams with intergenerational equity: The impossibility of being Paretian" *Econometrica* **71**, 1557–1563.

[7] Basu and Mitra (2007) consider a social welfare relation on $[0,1]^{\mathbb{N}}$, while we consider a social welfare relation on $\mathbb{R}^{\mathbb{N}}$. Basu and Mitra's (2007) characterization is valid under our domain condition.

[4] Basu, K., and T. Mitra (2007) "Utilitarianism for infinite utility streams: A new welfare criterion and its axiomatic characterization" *Journal of Economic Theory* **133**, 350–373.

[5] Blackorby, C., W. Bossert, and D. Donaldson (2002) "Utilitarianism and the Theory of Justice" in *Handbook of Social Choice and Welfare* Volume 1 by K.J. Arrow, A.K. Sen, and K. Suzumura, Eds, North-Holland, Amsterdam, 543–596.

[6] Bossert, W., Y. Sprumont, and K. Suzumura (2007) "Ordering infinite utility streams" *Journal of Economic Theory* **135**, 579–589.

[7] Campbell, D.E. (1985) "Impossibility theorems and infinite horizon planning" *Social Choice and Welfare* **2**, 283–293.

[8] Diamond, P. (1965) "The evaluation of infinite utility streams" *Econometrica* **33**, 170–177.

[9] Fleurbaey, M. and P. Michel (2001) "Transfer principles and inequality aversion, with an application to optimal growth" *Mathematical Social Sciences* **42**, 1–11.

[10] Fleurbaey, M. and P. Michel (2003) "Intertemporal equity and the extension of the Ramsey criterion" *Journal of Mathematical Economics* **39**, 777–802.

[11] Hara, C., T. Shinotsuka, K. Suzumura, and Y. Xu (2008) "Continuity and egalitarianism in the evaluation of infinite utility streams" *Social Choice and Welfare* **31**, 179–191.

[12] Kaneko, M., and K. Nakamura (1979) "The Nash social welfare function" *Econometrica* **47**, 423–435.

[13] Kamaga, K., and T. Kojima (2008a) "$\mathcal{Q}$-anonymous social welfare relations on infinite utility streams" *Social Choice and Welfare* (forthcoming).

[14] Kamaga, K., and T. Kojima (2008b) "On the leximin and utilitarian overtaking criteria with extended anonymity" 21COE-GLOPE Working Paper Series, Waseda University.

[15] Lauwers, L. (1997a) "Continuity and equity with infinite horizons" *Social Choice and Welfare* **14**, 345–356.

[16] Lauwers, L. (1997b) "Infinite utility: Insisting on strong monotonicity" *Australasian Journal of Philosophy* **75**, 222–233.

[17] Lauwers, L. (1997c) "Rawlsian equity and generalized utilitarianism with an infinite population" *Economic Theory* **9**, 143–150.

[18] Mitra, T., and K. Basu (2007) "On the Existence of Paretian Social Welfare Relations for Infinite Utility Streams with Extended Anonymity" in *Intergenerational Equity and Sustainability* by J. Roemer and K. Suzumura, Eds, Palgrave: London.

[19] Roberts, K.W.S. (1980) "Interpersonal comparability and social choice theory" *Review of Economic Studies* **47**, 421–439.

[20] Sakai, T. (2003a) "An axiomatic approach to intergenerational equity" *Social Choice and Welfare* **20**, 167–176.

[21] Sakai, T. (2003b) "Intergenerational preferences and sensitivity to the present" *Economics Bulletin* **4**, 1–6.

[22] Sakai, T. (2006) "Equitable intergenerational preferences on restricted domains" *Social Choice and Welfare* **27**, 41–54.

[23] Shinotsuka, T. (1997) "Equity, continuity, and myopia: A generalization of Diamond's impossibility theorem" *Social Choice and Welfare* **15**, 21–30.

[24] Szpilrajn, S. (1930) "Sur l'extension de l'ordre partiel" *Fundamenta Mathematicae* **16**, 386–389.