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On mixed and behavioural strategies

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Abstract

In the context of extensive form games, we are considering the relation between mixed and behavioural strategies. We look again at the famous Kuhn theorem and also discuss a result for games in which no path intersects any information set in more than one node. We apply the principle mathematical of induction on the number of information sets treated behaviourally.

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1. Introduction

Mixed strategies are defined to be probability distributions over pure strategies, while behavioural strategies attach a probability distribution to the moves from each information set. As we shall show below, in general there is no presumption that one type of strategy is superior, in reaching the nodes of the tree, to the other.

In a classic paper by Kuhn (1953), a relation was established between mixed and behavioural strategies for extensive form, finite games. It was shown that for games with at least two choices from each node, there is an equivalence between behavioural and mixed strategies if and only if the game is of perfect recall, i.e. such that no player forgets what he knew before. The significance of this result is that for such games it suffices to consider only behavioural strategies. This means that one can consider independently for every information set the probabilities attached to the choices from each of its nodes.

In this paper we consider once more, and extend, this result[1]. Discussions in the literature usually confine themselves either to the necessity part of Kuhn's theorem or simply reference his paper, explaining the significance of behavioural strategies.

Selten (1975) considers Kuhn's result to be important and in his Theorem 1 proves a version of the Kuhn's Theorem relating strategies to equivalent behavioural strategy mixtures. We feel that there is room for a further proof of the complete theorem. In this note, as in Ritzberger (2002), Kuhn's theorem is a consequence of combining two theorems, proofs of which are self contained. The new proofs provided in this note employ induction over information sets using the idea of an A-mixed strategy, a concept intermediate to behavioural and mixed strategies.

Furthermore we extend the investigation and show the superiority of mixed to behavioural strategies if and only if the game is of type $\mathcal{G}$, i.e. such that no path intersects any information set more than at one node[2]. Games of perfect recall obviously satisfy this condition although the converse is not necessarily true. Different types of games with imperfect recall. are discussed by Osborne - Rubinstein (1994).

From the outset, in his Definition 2, Kuhn requires a game to have not more than one vertex of an information set on any play of the game. So he does not address the relation between behavioural and mixed strategies for games of type $\mathcal{G}$, as we do in our Theorem 1.

2. Some Notation and Definitions

An *extensive form game* consists of

(i) A set $\mathcal{P}$ of *players*.

(ii) A tree $\mathcal{T}$.

(iii) A map t from $\mathcal{P}$ to the non-terminal nodes $\mathcal{N}$ of $\mathcal{T}$.

[1] After the completion and circulation of our work, it was pointed out to us that Ritzberger (2002) also proves Kuhn's theorem by combining two results. There is correspondence between his Theorem 3.2 and 3.3 and our Theorem 1 and 2, although the proofs of the results are different. Ritzberger follows Kuhn closely whereas we apply a mathematical induction on information sets. We believe that a Kuhn-type proof requires much mathematical notation and presents some difficulty in following all algebraic steps.

[2] Violation of this condition is what Piccione - Rubinstein (1997) define as 'absentmindedness'. Our use of the term 'game of type $\mathcal{G}$' corresponds to 'no-absentmindedness'. Piccione - Rubinstein (1997) prove a result related to Ritzberger's Theorem 3.2 and our Theorem 1. It is their Proposition 1 and is attributed by them to Isbell (1957).

For any $P \in \mathcal{P}$ with $N = t(P)$, the immediate successors of N are the actions available to P from N.
(iv) An equivalence relation ρ on $\mathcal{N}$ which is compatible with t and such that for $N_1 \rho N_2$ the actions available from N_1 and N_2 are in 1-1 correspondence. An equivalence class is a set of nodes owned by a single player which are indistinguishable for that player.
(v) For each terminal node a $\mathcal{P}$-indexed vector giving the payoff to each player on termination of the game at that node.

An *information set*, I, in a game tree is a set of non-terminal nodes belonging to a single player, P, such that the moves, (choices, actions), from any two of these nodes are identical. The intended interpretation of this concept is that the nodes of I are equivalent for P, in that he cannot distinguish between them. Each of them appears identical in admitting the same collection of possible choices.

Suppose a player, P, has k information sets $I_1, I_2, \ldots, I_k$. We will assume that there is more than one action available from any node of each I_i and denote the set of such actions available from I_i by Σ_i and let B_i be the set of probability measures on Σ_i.

Let $A \subseteq K = \{1, 2, \ldots, k\}$. An *A-mixed strategy*, (μ, b), is a probability distribution, μ, over

$$\Sigma_A = \prod_{i \in A} \Sigma_i$$

together with an independent element $b \in \prod_{j \notin A} B_j$.

In particular a K-mixed strategy is called a *mixed strategy* and it is, essentially, a probability distribution over Σ_K which we write just as Σ; a $\emptyset$-mixed strategy is called a *behavioural strategy* and we write $B = \prod_{j \in K} B_j$. A (directed) path connects in a unique manner the initial node of the tree with a terminal node.

Note. There is an intended crucial difference of interpretation between the two parts of an A-mixed strategy, (μ, b). The probabilities (given by μ) with which P will choose amongst the moves from any node in an information set $I_j : j \in A$ is determined before he enters that set: once the actual move is decided upon, it will be used whenever P finds himself in that set. But each time P finds himself within an information set $I_j : j \notin A$ he will choose what move to make according to the probability b_j.

Two strategies, $x,\ x'$, employed by Player P, of any of the above sort are said to be *equivalent* if, whatever strategies are adopted by the other players, the strategy profiles in which P employs either x or x' give the same probabilities of reaching each node of the tree. We will write this equivalence as $x \sim x'$.

For P, one set of strategies $\mathcal{C}$ is said to be *superior to* another set $\mathcal{D}$ if for all $\sigma' \in \mathcal{D}$ there exists $\sigma \in \mathcal{C}$ such that $\sigma \sim \sigma'$.

Suppose a path intersects $I_{i_1}, I_{i_2}, \ldots, I_{i_m}$ of P's information sets, possibly with repeats and not necessarily in that order, and requires move σ_{i_α} to be made from I_{i_α}. The probability that P will make his moves on this path is

$$Pr(\sigma_{i_1}, \ldots, \sigma_{i_m}) = Pr(\sigma_{i_1}) \times Pr(\sigma_{i_2} | \sigma_{i_1}) \times \ldots \times Pr(\sigma_{i_m} | \sigma_{i_1}, \ldots \sigma_{i_{m-1}}). \tag{1}$$

Note that generally we will use the notation $Pr(\sigma_{i_1}, \ldots, \sigma_{i_m})$ for a marginal probability when all the unmentioned events are summed out.

Suppose this is to be calculated for an A-mixed strategy (μ, b): we may assume w.l.o.g. that only the first p indices, $i_1, i_2, \ldots, i_p$ are in A. The factors from $p+1$ to m are then independent and given by

$$\prod_{j=p+1}^{m} b_{i_j}(\sigma_{i_j});$$

if $p = m$ this will be taken to be 1.

However, the probability of events described by the first p terms together equal $\mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p})$, which will be taken to be 1 if $p = 0$. Here we can have dependent behaviour. In particular, if $i_\alpha = i_\beta$ for $\alpha < \beta \leq p$ then the β^{th} factor is:

$$Pr(\sigma_{i_\beta}|\sigma_{i_1}, \sigma_{i_1}, \ldots, \sigma_{i_{\beta-1}}) = \begin{cases} 0 & \text{if} \quad \sigma_{i_\beta} \neq \sigma_{i_\alpha} \\ 1 & \text{if} \quad \sigma_{i_\beta} = \sigma_{i_\alpha}. \end{cases} \tag{2}$$

This distinction is only of significance if the game contains information sets which may be intersected by a path in more than one node, a possibility which we wish to exclude here. We thus make the following definitions:

Definition 1. A game is said to be *of type* $\mathcal{G}$ if no path intersects any information set in more than one node.

In such games the players know that they are in an information set only once. If they find themselves in an information set they know they have just entered it and that their next move will take them out of it.

A player P is said to have perfect recall if: whenever a node ν_1 in an information set I owned by P is preceded by a node ν_1' in an information set I' from which a move σ was made, and that was the last move made by P, then every $\nu_2 \in I$ is preceded by some $\nu_2' \in I'$ from which the move σ was made with no intervening moves by P.

Definition 2. A game is said to be *of perfect recall* if every player has perfect recall.

The intention of Definition 2 is that, if a player can recall all of his previous actions, he should still not be able to distinguish between the nodes of an information set I_2 by recalling how he got to that set. All he will recall from any node of that set is the same sequence of previous information sets and the moves he made from them.

The one-player game in Figure 1a shows why in Kuhn's theorem the assumption of at least two choices from each information set is needed. Namely in the case of Figure 1a we have equivalence between mixed and behavioural strategies although we have imperfect recall.

Figure 1b is a well known example of absentmindednees, and makes the point that mixed strategies need not be superior to behavioural strategies as it is the case for games with no-absentmindednees. Furthermore one cannot deduce that behavioural strategies are superior to mixed strategies, (see Glycopantis and Muir (1996)). The use of behavioural strategies implies that node B can now be reached while it was not available under mixed strategies. On the other hand it is impossible to reach nodes A, B, C with probabilities 1/2, 0, 1/2, using a behavioural strategy, while the mixed strategy, $(H,\ 1/2; V,\ 1/2)$ achieves this. It thus follows that neither behavioural nor mixed strategies are superior.

It is evident that any game of perfect recall is a game of type $\mathcal{G}$, but the converse is obviously not true.

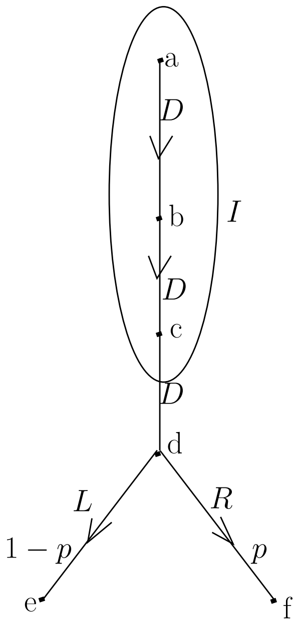


Figure 1a

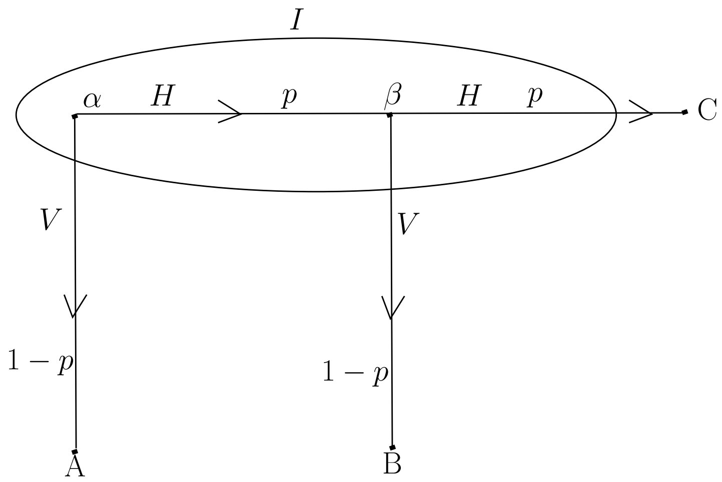


Figure 1b

Figure 1

3. Kuhn's Equivalence

Kuhn (1953) proved that under conditions of perfect recall for every mixed strategy there exists an equivalent behavioural strategy and that the converse is also true. Therefore it suffices to consider only behavioural strategies which concentrate on information sets as they are reached. Therefore the players act independently per information set and they do not to take into account their previous or later decisions.

First we present an example to see how this equivalence works.

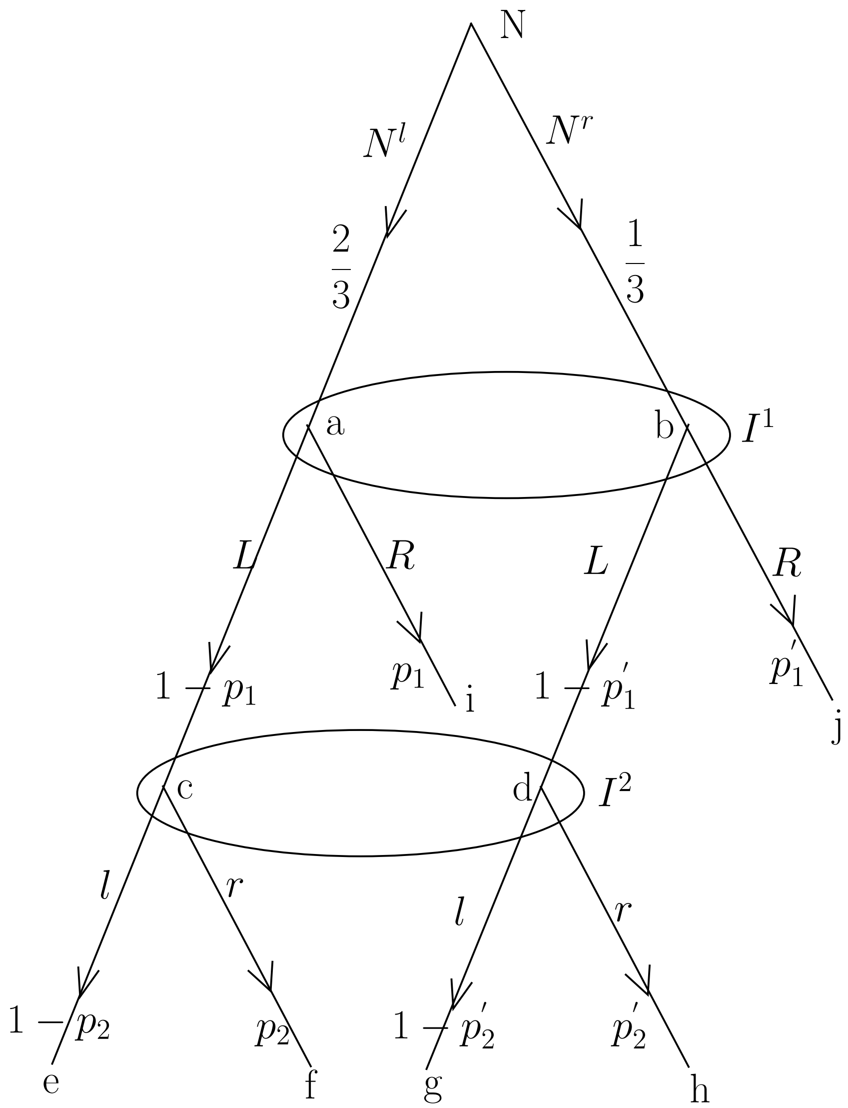
Example 1. Nature chooses N^l, N^r in the beginning with fixed probabilities. All other decisions are made by a single player as shown in Figure 2. The game has perfect recall. The player's pure strategies are $Ll, \quad Lr, \quad Rl, \quad Rr$ and we consider the general mixed strategy where these are played with corresponding probabilities $\pi_1, \quad \pi_2, \quad \pi_3, \quad \pi_4$. We want to find equivalent behavioural strategies by calculating the probabilities p_1, p_1', p_2, p_2' indicated on Figure 2 and showing that $p_1 = p_1'$ and $p_2 = p_2'$. We denote the probability of reaching node x from node y by $Pr(x|y)$.

We have

$$\begin{aligned} p_1 = Pr(i/a) &= Pr(N^l \cap R)/Pr(N^l) \\ &= Pr(N^l) \times Pr(R)/Pr(N^l) = Pr(R) \\ &= Pr(Rl) + Pr(Rr) = \pi_3 + \pi_4. \end{aligned}$$

We also have

$$\begin{aligned} p_1' = Pr(j/b) &= Pr(N^r \cap R)/Pr(N^r) \\ &= Pr(N^r) \times Pr(R)/Pr(N^r) = Pr(R) = \\ &Pr(Rl) + Pr(Rr) = \pi_3 + \pi_4. \end{aligned}$$



The player's pure strategies: *Ll, Lr, Rl, Rr*
Played with probabilities: $\pi_1,\ \pi_2,\ \pi_3,\ \pi_4$

Figure 2

Next

$$
\begin{aligned}
p_2 =& Pr(f|c) = Pr(N^l \cap L \cap r)/Pr(N^l \cap L) \\
&= Pr(N^l) \times Pr(L \cap r)/[Pr(N^l) \times Pr(L)] = Pr(L \cap r)/Pr(L) \\
&= Pr(Lr)/[Pr(Ll) + Pr(Lr)] = \frac{\pi_2}{\pi_1 + \pi_2},
\end{aligned}
$$

and finally

$$
\begin{aligned}
p_2' &= Pr(h|d) = Pr(N^r \cap L \cap r)/Pr(N^r \cap L) \\
&= Pr(N^r) \times Pr(L \cap r)/Pr(N^r) \times Pr(L) \\
&= Pr(L \cap r)/Pr(L) = \frac{\pi_2}{\pi_1 + \pi_2}.
\end{aligned}
$$

Throughout, $Pr(N^l), Pr(N^r)$, the probabilities of choice by nature, have been eliminated in the calculations.

It is perfect recall which guarantees here that we can replace the mixed by behavioural strategies.

So we have established that there exist equivalent probability distributions on the choices of the information sets I^1 and I^2. The question arises though about the property of independence between the distributions across the information sets. Independence must mean that we can choose these probabilities without reference to each other. On the other hand these distributions must be chosen in the above specific manner if they are to be equivalent to given mixed strategies.

First we consider conditions under which a behavioural strategy can be replaced by a mixed strategy.[3]

Theorem 1 A game is of type $\mathcal{G}$ if and only if, for any player, mixed strategies are superior to behavioural strategies.

Theorem 2 A game is of perfect recall if and only if, for any player, behavioural strategies are superior to mixed strategies.

Before embarking on the proof of these two theorems consider the path through any node ν of the tree which involves moves

$$(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m}) \in \Sigma_{i_1} \times \Sigma_{i_2} \times \ldots \times \Sigma_{i_m}$$

made by P. There will be some aggregate probability, π, for all the actions taken by the other players on the path to ν. P acts independently so the total probability of reaching ν is

$$\pi \times Pr\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m}\}, \tag{3}$$

where the second factor is to be calculated from the strategy employed by P. Since, by the definition of equivalence, all other players have fixed their chosen strategies (of whatever sort) checking the equivalence of any two strategies which may be employed by P amounts to showing the equality of the factor

$$Pr\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m}\}$$

for the two strategies.

The proof of the two theorems is based on an ordinary, finite induction the core of which shows the equivalence between an A_r-mixed strategy and an A_{r-1}-mixed strategy. In proving

[3]After formulating and proving Theorem I we discovered that part of its statement is implicit in an exercise in unpublished lecture notes by Binmore (1984).

Theorem 1, we go from an A_{r-1}-mixed strategy to an equivalent A_r-mixed strategy, by removing the behavioural status of one more information set and incorporating it into the mixed strategy component. An analogous, but reversed, argument is used in proving Theorem 2. In each case it must be shown that the given conditions allow any node of the tree to be reached with the same probability by either strategy.

Let $A_r = \{1, 2, \ldots, r\}$ for $r = 1, 2, \ldots, k$.

Proof of Theorem 1. Given $b = (b_1, b_2, \ldots, b_k) \in B$, consider the proposition

$$Q_r \stackrel{\text{def}}{=} \text{“There is an } A_r\text{-mixed strategy which is equivalent to } b\text{”}.$$

Clearly we are trying to establish Q_k. Q_1 holds since b itself is an A_1-mixed strategy, so we need only show that Q_{r-1} implies Q_r for $r = 2, 3, \ldots, k$.

Consider an A_{r-1}-mixed strategy, $(\hat{\mu}, \hat{b})$ where $\hat{\mu}$ is a probability distribution over $\Sigma_{A_{r-1}}$ and $\hat{b} = (b_r, b_{r+1}, \ldots, b_k) \in \prod_{j=r}^{k} B_j$. We show that the probability distribution μ over $\Sigma_{A_r} = \Sigma_{A_{r-1}} \times \Sigma_r$ given by the independent actions of $\hat{\mu}$ on $\Sigma_{A_{r-1}}$ and b_r on Σ_r, together with $b = (b_{r+1}, \ldots, b_k) \in \prod_{j=r+1}^{k} B_j$ forms an equivalent A_r-mixed strategy (μ, b).

Suppose a path goes through $I_{i_1}, I_{i_2}, \ldots, I_{i_m}$ (not necessarily in that order) only the first p of these indices being in A_{r-1}, that is $i_p < r$. The probability $Pr(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m})$ calculated by the A_{r-1}-mixed strategy $(\hat{\mu}, \hat{b})$ is

$$\hat{\mu}(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}) \times \prod_{j=p+1}^{m} b_{i_j}(\sigma_{i_j}).$$

We consider the two cases:

Case I(a). If $i_{p+1} \neq r$ the marginal of $\hat{\mu}$ will be identical to that of μ since in both cases we will get the marginal by summing over Σ_r.

Case I(b). If $i_{p+1} = r$, so that the path passes through I_r, the marginals $\hat{\mu}(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p})$ and $\mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p})$ will still be equal, since by hypothesis the game is of type $\mathcal{G}$ so I_r has not yet been reached; furthermore

$$b_r(\sigma_r) = \hat{\mu}(\sigma_r | \sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}).$$

We now have:

$$\begin{aligned}
&\mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}, \sigma_r) \times b(\sigma_{i_{p+2}}, \sigma_{i_{p+3}}, \ldots, \sigma_{i_m}) \\
&= \mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}) \times \mu(\sigma_r | \sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}) \times b(\sigma_{i_{p+2}}, \sigma_{i_{p+3}}, \ldots, \sigma_{i_m}) \\
&= \mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}) \times b_r(\sigma_r) \times \prod_{j=p+2}^{m} b_{i_j}(\sigma_{i_j}) \\
&= \hat{\mu}(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}) \times \hat{b}(\sigma_{i_{p+1}}, \ldots, \sigma_{i_m}).
\end{aligned} \tag{4}$$

Therefore the probability of the path reaching the specific node in I_r is the same under the A_{r-1}-mixed and A_r-mixed strategies.

For the converse, we show that if a game is not of type $\mathcal{G}$ then the stated equivalence does not hold. Suppose then that there is an information set I and a path which goes through I, first at node ν_1 and then at node ν_2. Consider corresponding moves L, R from those nodes. In any mixed strategy, we could not have move L made from ν_1 and R made from ν_2, whereas this is feasible in a behavioural strategy. We need only choose a behavioural strategy, b, in which the probabilities of L and R are non-zero. Figure 1b shows such an example, with a slight change in notation. This completes the proof of Theorem 1.

Proof of Theorem 2. Given a mixed strategy μ consider the proposition

$$S_q \stackrel{\text{def}}{=} \text{“There is an } A_{k-q+1}\text{-mixed strategy which is equivalent to } \mu\text{”.}$$

We are trying to establish S_k. S_1 holds since μ itself is an A_k-mixed strategy, so we need only show that S_q implies S_{q+1} for $q = 1, 2, \ldots, k$; or, equivalently, that for any A_r-mixed strategy there is an equivalent A_{r-1}-mixed strategy for $r = 2, 3, \ldots k$.

Suppose we are given an A_r-mixed strategy (μ, b) with μ on Σ_{A_r} and $b = (b_{r+1}, \ldots, b_k)$. We wish to form an equivalent pair $(\hat{\mu}, \hat{b})$ with $\hat{\mu}$ on $\Sigma_{A_{r-1}}$ and $\hat{b} \in \prod_{j=r}^{k} B_j$.

Suppose a path goes through $I_{i_1}, I_{i_2}, \ldots, I_{i_m}$ (not necessarily in that order) only the first p of these indices being in A_r, that is $i_p \leq r$. The probability $Pr(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m})$ calculated by the A_{r-1}-mixed strategy $(\hat{\mu}, \hat{b})$ is

$$\mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}) \times \prod_{j=p+1}^{m} b_{i_j}(\sigma_{i_j}).$$

We consider the two cases:

Case II(a). Considering the possibility that a path might not pass through I_r, we see that $\hat{\mu}$ must be the marginal of μ got by summing over Σ_r and the components of $\hat{b}$ in $\prod_{j=r+1}^{k} B_j$ will be identical with b. Which leaves only the construction of b_r. We can do this when consider a path through I_r.

Case II(b). Consider a path to a node $\hat{\nu}$ which passes through $I_{i_1}, I_{i_2}, \ldots, I_{i_m}$ (in that order), with $i_p = r$. Let the path pass through node ν in I_r: then the probability of reaching $\hat{\nu}$ is the probability of reaching $\hat{\nu}$ conditional on reaching ν times the probability of reaching ν, thus:

$$Pr(\hat{\nu}) = Pr(\hat{\nu}|\nu) \times Pr(\nu).$$

$Pr(\nu)$ arises from a path which does not involve I_r, since the assumption of perfect recall implies that a path cannot intersect an information set twice, so that factor will be the same for both the A_r-mixed strategy and the corresponding A_{r-1}-mixed strategy defined above.

$Pr(\hat{\nu}|\nu)$ may be re-written $Pr(\hat{\nu}|\bar{\nu}) \times Pr(\bar{\nu}|\nu)$ where $\bar{\nu}$ is the next node reached from ν using move σ_r from I_r. Now the first factor (which may be 1 if $\hat{\nu} = \bar{\nu}$) is the same for the two strategies since it only involves moves from information sets other than I_r. Theorem 2 is then completed if we can set

$$b_r(\sigma_r) = Pr(\bar{\nu}|\nu). \tag{5}$$

This is the point at which the full power of the assumption of perfect recall is needed. In general $b_r(\sigma_r)$ might not be well-defined by equation (5): the right-hand-side depends not just upon the move σ_r but upon the node ν from which that move was made. But we have

$$Pr(\bar{\nu}|\nu) = \frac{\mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_{p-1}}, \sigma_r)}{\mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_{p-1}})},$$

which is the same for any node of I_r since reaching such a node involves the same moves $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_{p-1}}$ by P.

We can now construct the analogue of relation (4) above

$$\begin{aligned}
&\mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_{p-1}}, \sigma_{r=i_p}) \times b(\sigma_{i_{p+1}}, \sigma_{i_{p+3}}, \ldots, \sigma_{i_m}) \\
=&\mu(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_{p-1}}) \times \mu(\sigma_r | \sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_{p-1}}) \times b(\sigma_{i_{p+1}}, \sigma_{i_{p+3}}, \ldots, \sigma_{i_m}) \\
=&\hat{\mu}(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_{p-1}}) \times \hat{b}(\sigma_{i_p}, \ldots, \sigma_{i_m}).
\end{aligned} \tag{6}$$

Therefore the probability of the path reaching the specific node in I_r is the same under the $\hat{A}$- and the A-mixed strategies.

To prove the converse consider two nodes ν_1, ν_2 in an information set I owned by P. Node ν_1 is preceded by node ν_1' in an information set I' from which P's last move was σ_1. We must consider a number of ways in which a player might fail to have perfect recall. These are illustrated in Figure 3 below. All information sets indicated belong to the same player. The interrupted lines indicate that on the path to a node other players, apart from the specific one we are considering, have also made decisions. Figure 3a is a general case of perfect recall.

Case 0. First we consider the case in which the game is not of type $\mathcal{G}$. Then we can consider strategies which make the game in Figure 1b relevant, for which we know that no behavioural strategies are equivalent to certain mixed strategies.

Case 1. $\nu_1' = \nu_2'$. This case is shown in Figure 3b. It is easy to show that for a particular mixed strategy there is no equivalent behavioural strategy.

Pure strategies of player P are $(\sigma_1\sigma,\ \sigma_1\neg\sigma,\ \sigma_2\sigma,\ \sigma_2\neg\sigma)$ and we assume that they are played with respective probabilities $(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ 0)$. This mixed strategy is not realisable by any behavioural strategy.

Case 2. ν_2' is in I' and this node is the last node owned by P on the path from ν_2' to ν_2, but the move made from ν_2' is $\sigma_2 \neq \sigma_1$, (see Figure 3c). We construct a mixed strategy which does not allow a consistent probability for the move σ from I.

Let the probabilities of moves from any information set other than I, I' be made independently of those from I, I'. We also choose that mixed strategy to have zero probability for any strategy which uses a move from I' other than σ_1, σ_2. The probability of P's moves on the path to ν_1 is $\mu(\sigma_1) \times \pi$ where π is the probability of P's moves on that path other than the move σ_1. The probability of P's moves on the path to $\bar{\nu}_1$ is $\mu(\sigma_1, \sigma) \times \pi$. Taking also into account the independent probabilities of the other players, including nature, we can have $\mu(\sigma_1) \times \pi'_1 + \mu(\sigma_2) \times \pi'_2 = 1$.

So

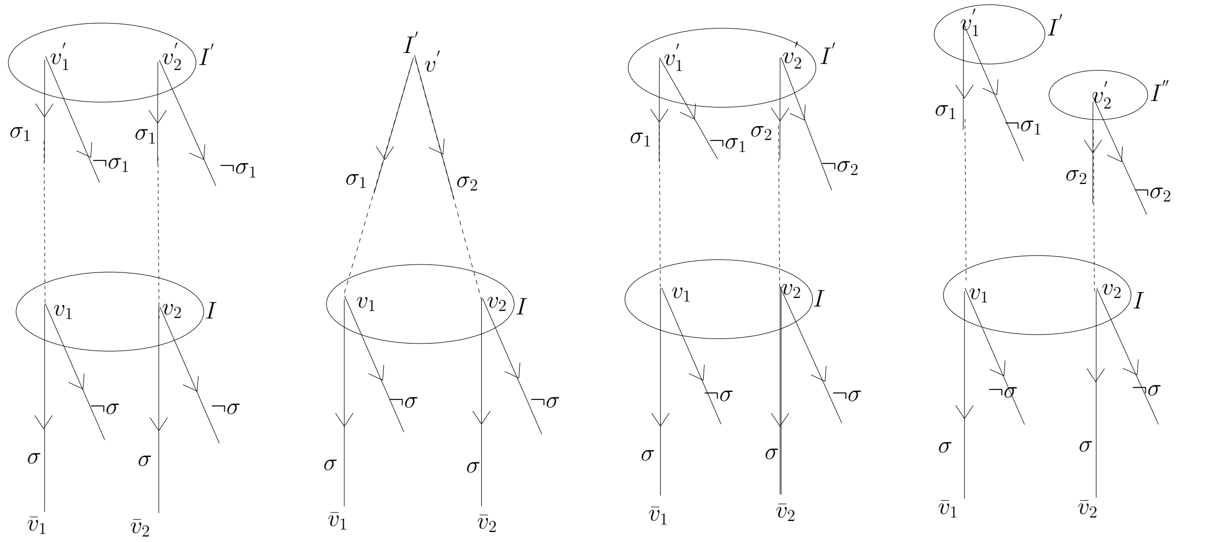
$$Pr(\bar{\nu}_1 | \nu_1) = \frac{\mu(\sigma_1, \sigma)}{\mu(\sigma_1)}.$$

Similarly

$$Pr(\bar{\nu}_2 | \nu_2) = \frac{\mu(\sigma_2, \sigma)}{\mu(\sigma_2)}.$$

Choose μ to have marginals

$$\mu(\sigma_1, \sigma) = \mu(\sigma_2, \sigma) = \mu(\sigma_1, \neg\sigma) = \frac{1}{3}, \ \ \mu(\sigma_2, \neg\sigma) = 0,$$



Perfect recall — Figure 3a

Imperfect recall: $v_1' = v_2' = v'$ — Figure 3b

Imperfect recall: $\sigma_1 \neq \sigma_2$ — Figure 3c

Imperfect recall: $I' \neq I''$ — Figure 3d

Figure 3

where $\neg\sigma$ is the event that σ is not used. Then

$$\mu(\sigma_1) = \mu(\sigma_1, \sigma) + \mu(\sigma_1, \neg\sigma) = \frac{2}{3},$$

$$\mu(\sigma_2) = \mu(\sigma_2, \sigma) + \mu(\sigma_2, \neg\sigma) = \frac{1}{3}.$$

So $Pr(\bar{\nu}_1|\nu_1) = \frac{1}{2}$ and $Pr(\bar{\nu}_2|\nu_2) = 1$. Thus there is no behavioural strategy giving a consistent probability for the move σ from I.

Case 3. This is shown in Figure 3d. The node ν_2' preceding ν_2 in P's choices lies in an information set $I'' \neq I'$.

As in Case 2 we are comparing $Pr(\bar{\nu}_i|\nu_i)$ for $i = 1, 2$.

We consider a mixed strategy for P, μ, in which probabilities of moves from any information set other than I, I' and $I''$and are made independently of those from I, I' and I''.

Assume the marginals concerning moves from these information sets to be

$$\mu(\sigma_1, \sigma, \sigma_2) = \mu(\sigma_1, \sigma, \neg\sigma_2) = \mu(\sigma_1, \neg\sigma, \sigma_2) = \frac{1}{3},$$

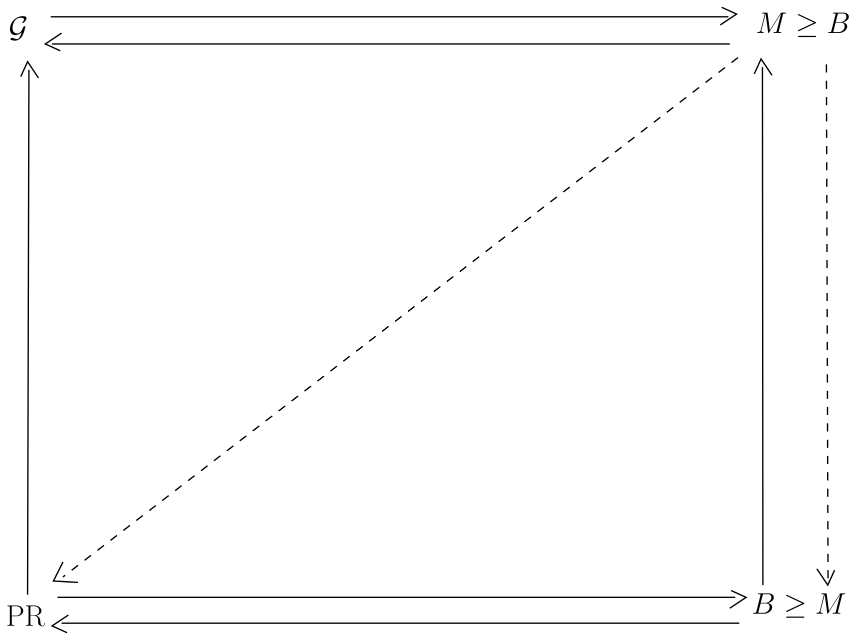
with all others being zero.

Then we have

$$\mu(\sigma_1, \sigma) = \mu(\sigma_1, \sigma, \sigma_2) + \mu(\sigma_1, \sigma, \neg\sigma_2) = \frac{2}{3}$$

$$\mu(\sigma_1, \neg\sigma) = \mu(\sigma_1, \neg\sigma, \sigma_2) + \mu(\sigma_1, \neg\sigma, \neg\sigma_2) = \frac{1}{3}$$

$$\mu(\sigma, \sigma_2) = \mu(\sigma_1, \sigma, \sigma_2) + \mu(\neg\sigma_1, \sigma, \sigma_2) = \frac{1}{3}$$



The relation between games of type $\mathcal{G}$, games with perfect recall (PR) , mixed (M) and behavioural strategies (B). The symbol $\geq$ means the left-side is superior to the right-hand side. A solid arrow means implication and an interrupted arrow denotes lack of implication.

Figure 4

$$\mu(\neg\sigma, \sigma_2) = \mu(\sigma_1, \neg\sigma, \sigma_2) + \mu(\neg\sigma_1, \neg\sigma, \sigma_2) = \frac{1}{3}.$$

Hence

$$\mu(\sigma_1) = \mu(\sigma_1, \sigma) + \mu(\sigma_1, \neg\sigma) = 1,$$
$$\mu(\sigma_2) = \mu(\sigma, \sigma_2) + \mu(\neg\sigma, \sigma_2) = \frac{2}{3}.$$

It follows that

$Pr(\bar{\nu}_1|\nu_1) = \frac{2}{3}$ and $Pr(\bar{\nu}_2|\nu_2) = \frac{1}{2}$. Thus there is no behavioural strategy giving a consistent probability for the move σ from I.

This completes the proof of Theorem 2.

Since games of perfect recall are also games of type $\mathcal{G}$, Theorems I and II combined together imply

Theorem 3 (Kuhn's theorem) A game is of perfect recall if and only if, for any player, behavioural strategies and mixed strategies are equivalent.

The discussion above can be summarized in terms of Figure 4. It indicates the relation between games with perfect recall (PR) or of type $\mathcal{G}$ and the possible superiority between mixed (M) and behavioural strategies (B). It should be read as follows. Heavy arrows indicate that games with PR imply (and are implied by) the superiority ($\geq$) of B over M. Games with of type $\mathcal{G}$ imply (and are implied by) the superiority ($\geq$) of M over B. We also have that the superiority of B over M implies the superiority of M over B.

On the other hand the superiority of M over B does not imply the superiority of B over M. This follows from the fact, shown by the one-player game in Figure 3b, that the superiority of M over B does not imply that the game is of PR.

4. Concluding Remarks

In this note we consider again the famous, and widely applied, classic Kuhn theorem which establishes that for extensive form, finite games with at least two choices from each node, there is an equivalence between behavioural and mixed strategies if and only if the game is of perfect recall, i.e. such that no player forgets what he knew before. The result is significant because it implies that for such games it suffices to consider only behavioural strategies.

We prove Kuhn's theorem by combining two results. In first, under no-absentmindness, mixed strategies are superior to behavioural strategies, and in the second, under perfect recall, which implies no-absentmindness, behavioural strategies are superior to mixed strategies. There is a correspondence between our results and those in Ritzberger (2002). However the proofs are different. Ritzberger follows Kuhn closely whereas we offer a new approach. We believe that a Kuhn-type proof requires much mathematical notation and presents some difficulty in following all algebraic steps. We prove the results by applying a mathematical induction argument on information sets.

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