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Revisiting Friendship Networks

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Abstract

We extend the model of friendship networks developed by Brueckner (2006) in two ways. First, we extend the level of indirect benefits by incorporating benefits from up to three links and explore its implication for the socially optimal and individual effort levels. Next, we generalize the magnetic agent problem by allowing for more than 3 players by restricting ourselves to regular networks that include payoffs from the magnetic agent.

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1 Introduction

Recent research in economics has shown that networks play an important role in determining the outcomes of social and economic relationships. These networks result from the interaction among economic agents. Our paper revisits a specific type of social network called friendship networks introduced in an interesting paper by Brueckner (2006). We extend the original paper by incorporating benefits from up to three links and examine the significance of these increased indirect benefits. We also add to Brueckner's paper by generalizing his magnetic agent model using a *regular* network structure. Thus the first extension focuses on the role of indirect benefits while the second extension generalizes the asymmetric benefits case for a fairly large class of networks.

There are two well-known approaches in economics that analyze network formation. The first of these was developed by Jackson and Wolinsky (1996) and involves mutual consent for the formation of networks. The equilibrium concept followed by this approach is called pairwise stability and can be thought of as a link based non-strategic concept. The second approach due to Bala and Goyal (2000a) uses the notion of Nash equilibrium. Note that the costs and benefits of links in the network in both these approaches are usually exogenously given.

Brueckner's model of friendship networks differs from the rest of the literature in the sense that the benefits from links depend on the effort exerted to create them, and therefore are not given exogenously. Consequently, to establish a friendship two individuals have to be acquainted beforehand and must exert effort to form this link. The model captures realism: after the two agents put in their effort, the success of every link is still a probabilistic event, where the probability depends on the effort put in by both individuals as well as a random error term. Thus link formation is stochastic. Also, link success is an independent event, i.e., the probability of link success between Alice and Bob is independent of the probability of link success between Bob and Carol. [1]

Analyzing the fully symmetric case where every player is acquainted with every other player, and all links yield equal benefits, Brueckner finds that if the equilibrium in the symmetric case is stable and unique, then the common equilibrium effort level is less than the socially optimal level. In this model agents obtain benefits from up to two links only (or level-one indirect benefits). He also examines the effect of asymmetry by including a "magnetic agent" who offers greater friendship benefits than the other non-magnetic individuals. By considering a model with only three agents he is able to show that when effort levels are substitutes, the nonmagnetic agents will exert more effort to form a friendship with magnetic agent than with the other (nonmagnetic) agents. The magnetic agent on the other hand will incur minimum effort to form a friendship with the nonmagnetic agents.

We extend Brueckner's basic model by considering the benefits from up to three links, i.e., benefits from friends of friends of friends (or level-two indirect benefits). Thus, in our model if there are three links we can say that an individual i gets benefit from being direct friend of j, and gets smaller benefit from indirect friendship with k who is a direct friend of j. He also gets benefits from l who is direct friend of k. The benefit i gets from l however is smaller than the benefit he gets from j and k. This framework allows us to investigate more generally the role of indirect benefits in the formation of friendship networks. Next, we generalize Brueckner's magnetic agent model (with only 3 individuals) by allowing for more than three players while restricting ourselves to regular networks that include payoffs from the magnetic agent.

[1] This is similar to the probabilistic Nash networks formulation of Bala and Goyal (2000b) and Haller and Sarangi (2005). A recent paper by Bloch and Dutta (2008) also has stochastic links whose strength is based on the level of effort exerted by the parties involved in the link. However, the set up and the issues pursued there differ substantially from Brueckner's paper.

The remainder of this paper is organized as follows. In Section 2 we set up the model and revisit the two main results of the original paper. First, we consider the case with benefits from three links followed by the generalization of the magnetic agent problem. Section 3 offers a summary of our results.

2 The Model

In this section we begin by developing the basic model. Unless explicitly stated otherwise, we follow the notation used in Brueckner (2006).

2.1 Model Setup

Denote the set of players in the game by $N = \{1, 2, \ldots, n\}$. For a generic player $i \in N$, we use $a(i)$ to denote the acquaintance set of player i. So $a(i) = \{j : i \text{ and } j \text{ are acquainted}\}$. Let e_{ij} denote the effort exerted by agent i to form a friendship with agent $j \in a(i)$. The probability that an ij friendship is formed depends on the effort exerted both by i and j, and is denoted by $P(e_{ij}, e_{ji})$ where $P(.,.) \in [0, 1)$. The friendship between i and j also depends on the realization of a random term. In other words, a friendship between i and j is established when $F(e_{ij}, e_{ji}) + \varepsilon > 0$, where F is an increasing function and ε is an error term that is identically distributed across all potential ij pairs. We assume that ε is independent across pairs. Then the probability of a successful friendship is $\text{Prob}[\varepsilon > -F(e_{ij}, e_{ji})] \equiv P(e_{ij}, e_{ji})$. We assume that this probabilistic friendship success function is symmetric, i.e., $P(e_{ij}, e_{ji}) = P(e_{ji}, e_{ij})$. Following Brueckner we also assume that the P function is increasing in both arguments with negative second partial derivatives. Hence $\partial^2 P/\partial e_{ij}^2 = \partial^2 P/\partial e_{ji}^2 < 0$.

Effort in this model is costly. For player i the effort required to establish an ij friendship costs $C(e_{ij})$ where C is strictly convex increasing function with $C(0) = 0$. Hence the total cost incurred by player i for all her friendship links is given by $\sum_{j \in a(i)} C(e_{ij})$. Although a number of different cost specifications are possible, here we adopt the one used by Brueckner (2006).

We now define the benefits from friendships in the model. Let $u_{ij} > 0$ and $v_{ik} > 0$ denote player i's benefits from a direct friend j and an indirect friend k respectively (obviously here j and k are direct friends). In our model we also take into account agent l who is a direct friend of agent k. Player i gets benefits equal to $w_{il} > 0$ from the sequence of links involving j, k and l. As before we assume that benefits decrease with distance giving us the following ranking: $u_{ij} > v_{ik} > w_{il}$. Finally, friendship benefits from different links are assumed to be cumulative.

2.2 Extending indirect benefits

We begin by computing the expected benefits of individual i from the friendship network

$$B_i = \sum_{j \in a(i)} P(e_{ij}, e_{ji}) \left[\begin{array}{c} \left\{ u_{ij} + \sum\limits_{\substack{k \in a(j) \\ k \neq i}} v_{ik} P(e_{jk}, e_{kj}) \right\} + \\ \sum\limits_{\substack{k \in a(j) \\ k \neq i}} P(e_{jk}, e_{kj}) \sum\limits_{\substack{l \in a(k) \\ l \neq i \neq j}} w_{il} P(e_{kl}, e_{lk}) \end{array} \right]. \tag{1}$$

In the above expression, the last term which captures the indirect benefits from the third player l is not present in Brueckner's formulation. Following the original paper we assume symmetry: friendship benefits are uniform across individuals and each person is initially acquainted with everyone else. This allows us to set $u_{ij} = u$, $v_{ij} = v$, $w_{ij} = w$ and $e_{ij} = e_3$ for all $i, j \in N \times N$. The subscript 3 in e_3 denotes the fact that we are dealing with the model of three links. Similarly let e_2 be the effort level when benefits are obtained from only two links. To obtain the optimal effort level we use the first order condition for the net expected benefit function. Using symmetry we get:

$$P^{'}(e_3, e_3)\left[u + (n-2)vP + (n-2)(n-3)wP^2\right] = C^{'}(e_3) \tag{2}$$

where $P^{'}(e_3, e_3)$ denotes the partial derivative with respect to the first argument and the multiplicative probability terms follow from the independence of the error term ε.

Next, we compute the socially optimal effort level. First let $\overline{W} = \sum_{i=1}^{n} B_i$ which can be written as

$$\overline{W} = \sum_{i \in N}\left[\sum_{j \in a(i)} P(e_{ij}, e_{ji})\left\{\begin{array}{c} (u_{ij} + u_{ji}) + \sum\limits_{\substack{k \in a(j)\\ k \neq i}} (v_{ik} + v_{ki})P(e_{jk}, e_{kj}) + \\ \sum\limits_{\substack{k \in a(j)\\ k \neq i}} P(e_{jk}, e_{kj})\left(\sum\limits_{\substack{l \in a(k)\\ l \neq i \neq j}} (w_{il} + w_{li})P(e_{kl}, e_{lk})\right)\end{array}\right\}\right]. \tag{3}$$

The social welfare function is just the sum of net (expected) individual benefits i.e.,

$$W = \overline{W} - \sum_{i=1}^{n}\sum_{j \in a(i)} C(e_{ij}). \tag{4}$$

Observe that in this function the planner takes into account the fact that greater effort by i raises expected direct friendship benefits for both i and j. Also note that the planner takes the externality from the indirect benefits of upto two levels into account. Invoking symmetry we can write the first order condition ($\frac{\partial W}{\partial e_{ij}} = 0$) as follows:

$$\begin{aligned} & P^{'}(e_3^*, e_3^*)\left[2u + 2(n-2)vP + 2(n-2)(n-3)wP^2\right] \; + \\ & P^{'}(e_3^*, e_3^*)\left[2(n-2)vP + 4(n-2)(n-3)wP^2\right] \\ = \; & C^{'}(e_3^*) \end{aligned} \tag{5}$$

which simplifies to[2]

$$2P^{'}(e_3^*, e_3^*)\left[u + 2(n-2)vP + 3(n-2)(n-3)wP^2\right] = C^{'}(e_3^*). \tag{6}$$

We can now state our first proposition.

PROPOSITION 1: *In the extended model, if the equilibrium in the symmetric case is unique and stable, then the common equilibrium effort level is smaller than the socially optimal level, with* $e_3 < e_3^*$. *Moreover* $e_2 < e_3$ and $e_2^* < e_3^*$.

Proof: See Appendix.

[2]Following Brueckner (2006) we assume that the Hessian is negative definite to ensure that the social welfare function can be maximized.

The intuition behind this result can be easily explained. Proposition 1 says that agents do not expend enough individual effort in forming friendship links. This is due to the presence of two different externalities in the model that are not taken into account by players when choosing their own effort level. The first externality stems from the fact that while forming links each individual ignores the reciprocal benefits enjoyed by those who become her direct and indirect friends. In other words, even if there were no indirect benefits, it is easy to see that $e_1 < e_1^*$, that is the individual effort choice is lower than the socially optimal one. The second externality arises due to the fact that each player neglects her role in facilitating indirect relationships between other people that make use of her own direct links. Hence as expected we find $e_3 < e_3^*$. This is similar to Brueckner's finding that $e_2 < e_2^*$. Next, it is also true that $e_2 < e_3$ and $e_2^* < e_3^*$. Due to the presence of the level-two indirect benefits agents spend greater effort than in case of level-one indirect benefits. This is true both for the individual as well as the social planner. When agents know they have access to more links, and thus more benefits than in the level-one indirect benefits case, they spend more effort to derive maximum benefits. Similarly, although the social planner takes the externalities into account, the additional benefit from the third link for all players also leads to her choosing a higher effort level than in the model with level-one indirect benefits. Finally, we believe that this phenomenon of higher effort levels for both the individual and the social planner will persist as we take higher levels of indirect benefits into account.

2.3 The magnetic agent problem for m-regular networks

We now solve the magnetic agent problem by allowing for more than three agents. The magnetic agent reflects asymmetry in benefits, i.e., it pays more to be connected directly or indirectly to the magnetic agent. Without loss of generality let agent 1 be the magnetic agent and let $x = 2, 3, \dots, n$ denote all the other (non-magnetic) agents. The direct benefit of linking to a magnetic agent is given by u_1 while the direct benefit of linking to a non-magnetic agent is given by u_x where $u_1 > u_x > 0$. Similarly, indirect benefit from the magnetic agent is worth v_1 while from the others it is worth v_x with $v_1 > v_x > 0$. In order to keep the analysis tractable, we now assume that $w_{il} = 0$ for all ordered pairs $il \in N \times N$. Hence in this part of the paper, agents derive benefits from only up to two links as in the original Brueckner (2006) paper.[3] Here, we restrict our analysis to regular networks. A network is said to be *regular* if every agents have the same number of neighbors. In an m-regular network every player has m direct neighbors, where $m \in [2, (n-1)]$.[4] Note that $m = (n-1)$ implies the *complete* network. Further, a network is said to be *connected* if there exists a path between any two agents i and j in the network.

In this model efforts required to establish a friendship link are assumed to be substitutes. Hence the P function is written as $P(e_{ij} + e_{ji})$ with $P' > 0$ and $P'' < 0$. Next, we assume that every agent has access to a magnetic agent. We denote by e_{x1} the effort exerted by the other agents to link to agent 1, and by e_{1x} the effort exerted by agent 1 to link to the others. The effort level of nonmagnetic agents when linking to one another is given by e_{xx}. Also, from the symmetry between all agents we set $\widetilde{e} = e_{1x} + e_{x1}$, $\widehat{e} = 2e_{xx}$.

The first order conditions are given by:

$$P^{'}(\widetilde{e})\left[u_x + (m-1)P(\widehat{e})v_x\right] = C^{'}(e_{1x}). \tag{7}$$

[3]We do not consider the knows-everyone agent formulation of Brueckner because the results in this case depend on the exact network architecture and therefore has to be dealt with case by case.

[4]In this paper we have ruled out 1−regular networks since these will not allow for indirect benefits.

$$P^{'}(\widetilde{e})\left[u_1 + (m-1)P(\widetilde{e})v_x\right] = C^{'}(e_{x1}). \tag{8}$$

$$P^{'}(\widehat{e})\left[u_x + P(\widetilde{e})v_1 + (m-2)P(\widehat{e})v_x\right] = C^{'}(e_{xx}). \tag{9}$$

The first equation shows the behavior of agent 1 when forming a link with the others. The second equation shows agent x forming a link with agent 1, whose $(m-1)$ links are with other agents. Equation (9) is for agent x connecting to another non-magnetic agent.

We now state our next proposition.

PROPOSITION 2: *Consider the set of m-regular networks where every agent has access to benefits from the magnetic agent.*[5] *Let $\widetilde{e} > \widehat{e}$. Then non-magnetic agents expend more effort attempting to link with agent 1 than she expends attempting to link with them. The non-magnetic agents expend an intermediate amount of effort in linking with one another. More precisely, $e_{x1} > e_{xx} \geq e_{1x}$.*

Proof: See Appendix.

Note there are some important differences between our framework and Brueckner's. Equations (7) and (8) above are very similar to equations (8) and (9) in his paper. Given that we assume a regular network with n players, Brueckner's equation (10) is replaced by equation (9) in our paper. Observe that equation (9) includes both $P(\widetilde{e})$ and $P(\widehat{e})$. In our model, this makes comparison between equations (8) and (9) difficult without additional assumptions.[6] Hence we assume that $\widetilde{e} > \widehat{e}$. Although it is a somewhat strong assumption, it allows us to provide general results for a large class of networks. Additionally, unlike Brueckner we do not need to assume $u_1 - u_x > v_1 - v_x$. Finally, note that the above proposition holds for the complete network since it is the $(n-1)$-regular network.

The intuition behind this result is also quite simple. Agent 1 provides both higher direct and indirect benefits. Hence non-magnetic agents have a higher incentive to link with agent 1 than with one another. On the other hand, agent 1 does not gain as much by linking to the others as they stand to gain from her. Hence her incentive to form friendship links is the least. Brueckner argues that it is difficult to generalize this result since adding more agents ($n > 3$) dilutes the importance of the magnetic agent. The precise network structure would determine whether the magnetic agent was important enough or not. We are able to get around this problem due to two reasons. By assuming $\widetilde{e} > \widehat{e}$ we are able to maintain the importance of the magnetic agent independent of the network structure. Also by focusing on regular networks, we are able to keep track of the exact amount of direct and indirect benefits of magnetic and non magnetic agents, independent of the size of the player set.

3 Conclusion

In this paper we extend the basic model of Brueckner (2006) by incorporating benefits from three links. Not surprisingly, we find that the individual optimal effort level is lower than the social optimum. What is more interesting is the fact that the individual's optimal effort choice with benefits from three links exceeds the individual's effort choice with benefits from two links. The same is true for the socially optimal effort level. Next, we are also able to generalize the magnetic agent setup of Brueckner (2006) for m-regular networks that always include benefits from the magnetic agent.

[5] For this proof it is not enough to consider *connected* m-regular networks. It is important that the non-magnetic agents have access to benefits from the magnetic agent.

[6] Note that in Brueckner's paper the analogous equations involve only a $P(\widetilde{e})$ term.

4 Appendix

1. **Proof of Proposition 1***:* We can rewrite the first order condition for the individual's problem as follows:

$$\Phi_3(e_3) = \left[P^{'}(e_3, e_3)\{u + (n-2)vP + (n-2)(n-3)wP^2\} - C^{'}(e_3)\right] = 0.$$

where for local stability we assume that $\Phi^{'}(e) < 0$. Similarly we can rewrite the first order condition for the planner's problem as follows:

$$\Gamma_3(e_3^*) = 2P^{'}(e_3^*, e_3^*)\left[u + 2(n-2)vP + 3(n-2)(n-3)wP^2\right] - C^{'}(e_3^*) = 0.$$

Comparing these two we can see that $\Gamma_3(e_3^*) > \Phi_3(e_3^*)$. However using the fact that $\Gamma_3(e_3^*) = 0$, we can conclude that $\Phi_3(e_3^*) < 0$. Next with the stability of the equilibrium implying $\Phi^{'}(e) < 0$ and uniqueness implying that $\Phi(e)$ has a single solution, it follows from $\Phi_3(e_3^*) < 0$ that the solution must satisfy $e_3 < e_3^*$. This completes the proof of the first part. The first order condition for the two link benefits case is given by

$$P^{'}(e_2, e_2)\left[u + (n-2)vP\right] = C^{'}(e_2)$$

Rewriting this we get

$$\Phi_2(e_2) = \left[P^{'}(e_2, e_2)\{u + (n-2)vP\} - C^{'}(e_2)\right] = 0.$$

Comparing $\Phi_3(e)$ with $\Phi_2(e)$ we find $\Phi_3(e) > \Phi_2(e)$ for all values of e. In particular at $e = e_2$, $\Phi_3(e_2) > \Phi_2(e_2) = 0$. Therefore $\Phi_3(e_2) > 0$. But $\Phi_3(e_3) = 0$. The stability and uniqueness of the equilibrium yields $e_3 > e_2$. This proves the second part of our proposition. The first order condition for the social planner's problem for the two link benefits case is given below:

$$\Gamma_2(e_2^*) = \left[2P^{'}(e_2^*, e_2^*)\{u + 2(n-2)vP\} - C^{'}(e_2^*)\right] = 0.$$

Evaluating $\Gamma_3(e)$ at $e = e_2^*$ and comparing with $\Gamma_2(e_2^*)$, we get that $\Gamma_3(e_2^*) > \Gamma_2(e_2^*) = 0$. If $\Gamma' < 0$, and since $\Gamma_3(e_3^*) = 0$, the stability of the equilibrium implies $e_3^* > e_2^*$. To sum up, we have $e_3 < e_3^*$, $e_2 < e_3$ and $e_2^* < e_3^*$.

2. **Proof of Proposition 2***:* We will first show that $e_{x1} > e_{1x}$. Suppose not. Then $e_{1x} \geq e_{x1}$ from which it follows that $C^{'}(e_{1x}) \geq C^{'}(e_{x1})$. But this implies that

$$P^{'}(\widetilde{e})\left[u_x + (m-1)P(\widehat{e})v_x\right] \geq P^{'}(\widetilde{e})\left[u_1 + (m-1)P(\widetilde{e})v_x\right].$$

Given that $u_1 > u_x$, this can only happen if $\widehat{e} > \widetilde{e}$ which violates the assumption stated in the proposition. Hence we have a contradiction and it follows that $e_{x1} > e_{1x}$.

Next, we will show that $e_{xx} \geq e_{1x}$. Here we will assume $m \geq 3$, though it is easy to check that the proof is also valid for $m = 2$. In order to prove this part, lets assume the contrary. Therefore $e_{1x} > e_{xx}$ and hence $C^{'}(e_{1x}) > C^{'}(e_{xx})$. So,

$$P^{'}(\widetilde{e})\left[u_x + (m-1)P(\widehat{e})v_x\right] > P^{'}(\widehat{e})\left[u_x + P(\widetilde{e})v_1 + (m-2)P(\widehat{e})v_x\right]$$

However, this inequality will not hold since $v_1 > v_x$ and $\widetilde{e} > \widehat{e}$ by assumption. Hence it follows that $e_{xx} \geq e_{1x}$.

Next we look at the relationship between e_{x1} and e_{xx}. Since $\widetilde{e} > \widehat{e}$ and $e_{x1} > e_{1x}$, it is not possible to have $e_{xx} \geq e_{x1}$. Hence $e_{x1} > e_{xx}$.

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