Economics Bulletin

Volume 30, Issue 1

Long-Run Impacts of Inflation Tax with Endogenous Capital Depreciation

Seiya Fujisaki
Graduate School of Economics, Osaka University

Kazuo Mino
Institute of Economic Research, Kyoto University

Abstract

This paper examines the long-run impact of inflation tax in the context of a generalized Ak growth model in which the rate of capital depreciation is endogenously determined. We assume that the rate of capital depreciation positively depends on capital utilization rate and negatively depends on maintenance expenditures. Money is introduced via a cash-in-advance constraint that may apply to the maintenance expenditures as well as to consumption and investment spendings. We find that the long-run effects of inflation tax are more complex than those obtained in the monetary Ak growth model with a fixed capital depreciation rate. In particular, the relation between inflation and growth is highly sensitive to the specification of the capital depreciation technology as well as to the forms of cash-in-advance constraints.

Citation: Seiya Fujisaki and Kazuo Mino, (2010) "Long-Run Impacts of Inflation Tax with Endogenous Capital Depreciation", *Economics Bulletin*, Vol. 30 no.1 pp. 808-816.
Submitted: Jul 22 2009. **Published:** March 25, 2010.

1 Introduction

It has been claimed that the activity of maintaining and repairing equipment and structures is large relative to investment and it would be a substantial substitute with new investment.[1] Considering this fact, several authors introduce maintenance costs and endogenous capital depreciation into the standard models of growth and business cycles: see Aznar-Màrquez and Ruiz-Tamarit (2004), Guo and Lansing (2007), Licandro and Puch (2000), Licandro, Puch and Ruiz-Tamarit (2001) and McGrattan and Schmitz Jr. (1999). These studies show that introducing maintenance expenditures may alter both dynamic behavior and the stationary-state characterization of the model economy in a substantial manner.

The purpose of this paper is to explore the long-run impacts of inflation tax in the context of a generalized Ak growth model in which the rate of capital depreciation is endogenously determined. Following the existing literature mentioned above, we assume that the capital depreciation rate positively depends on the rate of capital utilization and negatively depends on maintenance expenditures. Money is introduced via a cash-in-advance constraint that may apply to the maintenance spendings as well as to consumption and investment expenditures. We find that the long-run effects of inflation tax are more complex than those obtained in the monetary Ak growth model with a fixed capital depreciation rate.[2] In particular, the relation between inflation and growth is highly sensitive to the specification of the capital depreciation technology as well as to the forms of cash-in-advance constraints.

2 Model

We assume that the rate of capital depreciation, δ, depends positively on the rate of capital utilization, u, and negatively on maintenance expenditures per capital stock, z/k :

$$\delta = \delta\left(u, \frac{z}{k}\right), \quad \delta_1 > 0, \quad \delta_2 < 0, \tag{1}$$

where z denotes maintenance expenditures and k is capital stock. To ensure the second-order conditions for the optimization problem shown below, we assume that function $\delta\left(u, z/k\right)$ is strictly convex in u and z/k. The production technology is given by an Ak production function such that

$$y = Auk, \quad A > 0, \tag{2}$$

where y denotes aggregate output. Namely, the ratio of output and the utilized capital is fixed.

We consider a competitive, representative-agent economy. The optimization problem for the representative household is given by the following:

$$\max \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} dt, \quad \rho > 0, \quad \sigma > 0$$

[1]See McGrattan and Schmitz Jr. (1999) and Mullen and Williams (2004).

[2]The standard Ak growth model with cash-in-advance constraints are studied by Chen and Guo (2008a, b), Jha, Yip and Wang (2002), Li and Yip (2004) and Suen and Yip (2005).

subject to

$$\dot{m} = y - c - v - z - \pi m + \tau, \tag{3}$$

$$\dot{k} = v - \delta k, \tag{4}$$

$$c + \phi_1 v + \phi_2 z \leq m, \quad 0 \leq \phi_1, \ \phi_2 \leq 1, \tag{5}$$

together with the initial holdings of m and k. Here, c denotes consumption, m real money balances, v investment spending, π rate of inflation, and τ is a lump-sum transfer (lump-sum tax if it is negative) from the government. In addition, the household's income y and the capital depreciation rate δ are given by (1) and (2), respectively. In this problem, (3) is the flow budget constraint for the household, (4) describes capital formation and (5) specifies the cash-in-advance constraint. We assume that the cash constraint is applied to the entire consumption expenditure as well as to parts of maintenance and investment spendings.

The current-value Hamiltonian function for the household's optimization problem is given by

$$\begin{aligned} H \;=\; & \frac{c^{1-\sigma}-1}{1-\sigma} + q\left(Auk - c - v - z - \pi m + \tau\right) + \lambda\left[v - \delta\left(u, \frac{z}{k}\right)k\right] \\ & + \theta\left(m - c - \phi_1 v - \phi_2 z\right), \end{aligned}$$

where q and λ respectively denote the implicit prices of m and k, and θ is a Lagrangian multiplier. The control variables in this problem are c, u, v and z, while the state variables are m and k. The necessary conditions for an optimum are the following:

$$c^{-\sigma} - q - \theta = 0, \tag{6}$$

$$-q + \lambda - \theta\phi_1 = 0, \tag{7}$$

$$qA - \lambda\delta_1\left(u, \frac{z}{k}\right) = 0, \tag{8}$$

$$-q - \lambda\delta_2\left(u, \frac{z}{k}\right) - \theta\phi_2 = 0, \tag{9}$$

$$\dot{q} = q\left(\rho + \pi\right) - \theta, \tag{10}$$

$$\dot{\lambda} = \lambda\left[\rho + \delta\left(u, \frac{z}{k}\right) - \delta_2\left(u, \frac{z}{k}\right)\frac{z}{k}\right] - qAu, \tag{11}$$

$$\theta\left(m - c - \phi_1 v - \phi_2 z\right) = 0, \quad \theta \geq 0, \ \left(m - c - \phi_1 v - \phi_2 z\right) \geq 0, \tag{12}$$

along with (3), (4), the initial conditions and the transversality conditions:

$$\lim_{t\to\infty} qme^{-\rho t} = 0; \qquad \lim_{t\to\infty} \lambda ke^{-\rho t} = 0.$$

Note that (12) displays the Kuhn-Tucker conditions for the cash-in-advance constraint.

The market equilibrium condition for the final goods is

$$y = c + v + z \tag{13}$$

and the real money balances change according to

$$\dot{m} = m\left(\mu - \pi\right), \tag{14}$$

where μ denotes a given growth rate of nominal money stock. We assume that there is neither public debt nor the government's spending, so that a newly created money is distributed to the household as a lump-sum transfer. Hence, the government's flow budget constraint is given by $\mu m = \tau$.

3 Balanced-Growth Characterization

Given our specification of the model economy, it is easy to see that the balanced-growth equilibrium are characterized by the following conditions:

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{y}}{y} = \frac{\dot{v}}{v} = \frac{\dot{z}}{z} = \frac{\dot{m}}{m} = g, \tag{15}$$

$$\frac{\dot{q}}{q} = \frac{\dot{\lambda}}{\lambda} = \frac{\dot{\theta}}{\theta} = \gamma, \tag{16}$$

where g and γ are common growth rates that are constant over time on the balanced-growth path. As a result, the balanced-growth equilibrium requires that the capital utilization rate, u, and the maintenance expenditures per capital, z/k, are also constant.

First, it is to be noted that conditions in (15) and (16) mean the following:

$$\gamma = -\sigma g, \tag{17}$$

$$\pi = \mu - g. \tag{18}$$

Equation (17) comes from (6) , and (18) is given by (14). Condition (7) yields

$$\frac{\theta}{\lambda} = \frac{1}{\phi_1}\left(1 - \frac{q}{\lambda}\right). \tag{19}$$

Using (19) and (9) , we obtain

$$\frac{q}{\lambda} = \frac{\phi_1 \delta_2 (u, x) + \phi_2}{\phi_2 - \phi_1}, \tag{20}$$

where $x = z/k$. Consequently, (8) and (20) give

$$\delta_1 (u, x) = A \frac{\phi_1 \delta_2 (u, x) + \phi_2}{\phi_2 - \phi_1} \tag{21}$$

This equation represents the relationship between the optimal levels of capital utilization rate, u, and the maintenance spending rate, $x\,(= z/k)$. Notice that since we have not used the balanced-growth conditions to derive (21) , this relation also holds out of the balanced growth path.

From (10), (19) and (20) , we obtain

$$\begin{aligned} \frac{\dot{q}}{q} &= \rho + \pi - \frac{1}{\phi_1}\left(\frac{\lambda}{q} - 1\right) \\ &= \rho + \pi - \frac{1}{\phi_1}\left[\frac{\phi_2 - \phi_1}{\phi_1 \delta_2 (u, x) + \phi_2} - 1\right]. \end{aligned} \tag{22}$$

Using (17), (18) and (21) , we see that (22) is rewritten as

$$g = \frac{1}{1 - \sigma}\left\{\rho + \mu - \frac{1}{\phi_1}\left[\frac{A}{\delta_1 (u, x)} - 1\right]\right\}. \tag{23}$$

Similarly, (11) and (20) yield:

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta\left(u, x\right) - \delta_2\left(u, x\right) x - \left[\frac{\phi_1 \delta_2\left(u, x\right) + \phi_2}{\phi_2 - \phi_1}\right] Au.$$

In view of (17) and (21), the above equation is expressed as

$$g = \frac{1}{\sigma}\left\{\delta_1\left(u, x\right) u - \rho - \delta\left(u, x\right) + \delta_2\left(u, x\right) x\right\}. \tag{24}$$

To sum up, (21), (23) and (24) may determine the steady-state levels of x, u and g.

4 Long-Run Impacts of Inflation Tax

To clarify our analysis, we now specify the depreciation function as

$$\delta\left(u, x\right) = \frac{\delta_0 u^{\varepsilon}}{1 + \beta x}, \quad \varepsilon > 1, \quad \beta > 0, \quad \delta_0 > 0.$$

In the above, the capital depreciation rate is fixed at δ_0 when $\varepsilon = \beta = 0$. Our specification is a slightly modified version of the depreciation function used by Licandro and Puch (2000) and Guo and Lansing (2008). Given the above functional form, the steady-state conditions (21), (23) and (24) are respectively expressed by the following:

$$\frac{\varepsilon \delta_0 u^{\varepsilon - 1}}{1 + \beta x} = \frac{A}{\phi_2 - \phi_1}\left[\phi_2 - \phi_1 \frac{\beta \delta_0 u^{\varepsilon}}{(1 + \beta x)^2}\right], \tag{25}$$

$$g = \frac{1}{\sigma - 1}\left\{\frac{1}{\phi_1}\left[\frac{A\left(1 + \beta x\right)}{\varepsilon \delta_0 u^{\varepsilon - 1}} - 1\right] - \rho - \mu\right\}, \tag{26}$$

$$g = \frac{1}{\sigma}\left[\left(\varepsilon - 1 - \frac{\beta}{1 + \beta x}\right)\frac{\delta_0 u^{\varepsilon}}{1 + \beta x} - \rho\right]. \tag{27}$$

We examine the effects of a change in the money growth rate, μ, on the balanced growth path under alternative forms of the cash-in-advance constraint.

Before analyzing the above conditions, it is worth remembering the main findings in the standard *Ak* growth model with a constant capital deprecation rate. In the models with fixed depreciation, it is shown that the balanced-growth path is uniquely determined and a rise in the growth rate of money supply depresses the balanced-growth rate, as long as the elasticity of intertemporal substitutability in consumption, $1/\sigma$, is less than one.[3] In contrast, if $1/\sigma > 1$, then there may exist dual balanced-growth paths and a higher money growth rate raises the growth rate of income on the balanced-growth path with a higher growth rate.

We find that when $1/\sigma > 1$, multiple balanced-growth paths may emerge in our model as well. To emphasize the effects of endogenizing capital depreciation, in what follows, we focus on the case where $1/\sigma < 1$.

[3] See, for example, Chen and Guo (2008b), Li and Yip (2004) and Suen and Yip (2005).

Case (i): $\phi_1 = \phi_2 = 0$

First, we assume that the cash-in-advance constraint binds consumption expenditures alone. In this case $q = \lambda$ for all $t \geq 0$, and thus (8) and (9) respectively become $\varepsilon\delta_0 u^{\varepsilon-1} = A(1+\beta x)$ and $\beta\delta_0 u^{\varepsilon} = (1+\beta x)^2$. These equations give

$$u = \frac{\varepsilon}{\beta A}(1+\beta x), \tag{28}$$

implying that the optimal level of capital utilization rate is proportional to the optimal rate of maintenance spending. Using (8) and (28), we see that the optimal levels of x and u are given by

$$u^* = (\beta\delta_0)^{\frac{1}{2-\varepsilon}}\left(\frac{\varepsilon}{\beta A}\right)^{\frac{2}{2-\varepsilon}}, \qquad x^* = \frac{1}{\beta}\left[\left(\beta\delta_0\left(\frac{\varepsilon}{\beta A}\right)^{\varepsilon}\right)^{\frac{1}{2-\varepsilon}} - 1\right]. \tag{29}$$

Hence, the capital utilization and maintenance spending rates (so the capital depreciation rate) stay constant even out of the balanced-growth equilibrium.

The balanced-growth rate is determined as

$$g = \frac{1}{\sigma}\left[\left(\varepsilon - 1 - \frac{\beta}{1+\beta x^*}\right)\frac{\delta_0 u^{*\varepsilon}}{1+\beta x^*} - \rho\right],$$

where u^* and x^* are given by (29). As well as in the standard Ak growth model with the cash-in-advance constraint, our model shows that money is superneutral as to the balanced growth rate when the cash-in-advance constraint applies to consumption alone.

Case (ii): $\phi_1 > 0$ and $\phi_2 = 0$

Suppose that the maintenance expenditures are free from the cash-in-advance constraint. As in Case (i), if $\phi_2 = 0$, equation (28) always holds. Note that from (7) q generally diverges from λ. Hence, plugging (28) into (26) and (27), we obtain the following:

$$g = \frac{1}{\sigma - 1}\left\{\frac{1}{\phi_1}\left[\frac{A}{\varepsilon\delta_0}\left(\frac{\varepsilon}{\beta A}\right)^{1-\varepsilon}(1+\beta x)^{2-\varepsilon} - 1\right] - \rho - \mu\right\}, \tag{30}$$

$$g = \frac{1}{\sigma}\left[\left(\varepsilon - 1 - \frac{\beta}{1+\beta x}\right)\delta_0\left(\frac{\varepsilon}{\beta A}\right)^{\varepsilon}(1+\beta x)^{\varepsilon-1} - \rho\right]. \tag{31}$$

Remember that we have assumed that $\sigma > 1$ and $\varepsilon > 1$. Under these restrictions, we see that if $\varepsilon > 2$, then the graph of (30) has a negative slope and that of (31) has a positive slope. Therefore, there exists a unique balanced-growth path. It is also easy to see that a rise in money growth rate, μ, shifts down the graph of (30), so that a rise in μ lowers g and u. As a result, from (28) both x and δ decrease as well. In contrast, if $1 < \varepsilon < 2$, then both graphs have positive slopes. This means that these graphs may have multiple intersections.[4] Furthermore, if the graph of (30) is steeper than that

[4]When $x = 0$, equations (30) and (31) respectively give:

$$g_1|_{x=0} = \frac{1}{\sigma - 1}\left\{\frac{1}{\phi_1}\left[\frac{A}{\varepsilon\delta_0}\left(\frac{\varepsilon}{\beta A}\right)^{1-\varepsilon} - 1\right] - \rho - \mu\right\},$$

of (31) at an intersection, then a downward shift of the locus of (30) yields simultaneous increases in u, x and g. In this case, we obtain a positive long-run relation between money growth and the growth rate of real income.

Case (iii): $\phi_1 = 0$ *and* $\phi_2 > 0$

In this case, the cash-in-advance constraint does not apply to investment but it binds the maintenance expenditures. Condition $\phi_1 = 0$ means that $q = \lambda$ for all $t \geq 0$. Thus (8) becomes $\varepsilon\delta_0 u^{\varepsilon-1} = A(1+\beta x)$, which yields the relation between x and u in such a way that

$$u = \left(\frac{A}{\varepsilon\delta_0}\right)^{\frac{1}{\varepsilon-1}} (1+\beta x)^{\frac{1}{\varepsilon-1}}. \tag{32}$$

It is assumed that $\phi_2 > 0$ and thus (32) presents

$$\frac{\theta}{q} = \frac{1}{\phi_2}\left[\frac{\beta\delta_0 u^{\varepsilon}}{(1+\beta x)^2} - 1\right]. \tag{33}$$

From (10), (18) and (33), the balanced-growth relation (26) in the general case is replaced with

$$g = \frac{1}{\sigma-1}\left\{\frac{1}{\phi_2}\left[\beta\delta_0\left(\frac{A}{\varepsilon\delta_0}\right)^{\frac{\varepsilon}{\varepsilon-1}}(1+\beta x)^{\frac{2-\varepsilon}{\varepsilon-1}} - 1\right] - \rho - \mu\right\}. \tag{34}$$

Additionally, by use of (32), we write (27) as

$$g = \frac{1}{\sigma}\left[\left(\varepsilon - 1 + \frac{\beta}{1+\beta x}\right)\delta_0\left(\frac{A}{\varepsilon\delta_0}\right)^{\frac{\varepsilon}{\varepsilon-1}}(1+\beta x)^{\frac{1}{\varepsilon-1}} - \rho\right]. \tag{35}$$

Equations (34) and (35) demonstrate that the comparative statics results are similar to those in Case (ii): again, if $\varepsilon > 2$, then the balanced-growth path is uniquely determined and a rise in μ depresses x, u and g. If $1 < \varepsilon < 2$, a higher μ may increase x, u and g. on the balanced growth path.

Case (iv): $0 < \phi_1 \leq 1$ *and* $0 < \phi_2 \leq 1$

As the special cases mentioned above suggest, if neither ϕ_1 nor ϕ_2 is zero, we may have a variety of comparative statics results on the balanced growth path. Notice that (25) is written as

$$\left(1 - \frac{\phi_1}{\phi_2}\right)\frac{\varepsilon\delta_0 u^{\varepsilon-1}}{1+\beta x} + \frac{\phi_1}{\phi_2}\frac{\beta\delta_0 u^{\varepsilon}}{(1+\beta x)^2} = A.$$

Hence, if $\phi_2 > \phi_1$, then x and u satisfying the above equation change in the same direction, implying that the above gives $x = x(u)$, $x' > 0$. Substituting this into (26) and (27), we obtain the relations

---

$$g_2|_{x=0} = \frac{1}{\sigma}\left[(\varepsilon - 1 - \beta)\delta_0\left(\frac{\varepsilon}{\beta A}\right)^{\varepsilon} - \rho\right].$$

If ε is close to 2.0, the graph of (30) has strong concavity, while that of (31) is almost straight line. Thus when $g_1|_{x=0} < 0 < g_2|_{x=0}$, the both graphs would intersect twice. An increase in μ makes the graph of (30) shifts down, implying that the growth rate in the low-growth balanced growth path will increase.

between g and u that are similar to (27) and (30) (or (34) and (35)). Thus the effects of a change in μ will be the same as those in Cases (ii) and (iii). If $\phi_2 < \phi_1$, it is possible that (25) yields a negative relation between u and x. If this is the case, the comparative statics exercises become more complex than in Cases (ii) and (iii), even if assume that $1/\sigma < 1$.

Finally, consider the case where the same degree of cash constraint is apply for new investment and maintenance expenditures. Namely, we assume that $0 < \phi_1 = \phi_2 = \phi < 1$. Given this condition, the optimization condition as to the choice of x (Equation (9)) becomes $\delta_2(u, x) = -1$. Using our specification, we find that $\delta_2(u, x) = -1$ gives

$$1 + \beta x = (\beta \delta_0)^{1/2} u^{\varepsilon/2}. \tag{36}$$

Substituting (36) into (26) and (27), we obtain the following pair of equations:

$$g = \frac{1}{\sigma - 1}\left\{\frac{1}{\phi}\left[\frac{A}{\varepsilon}\left(\frac{\beta}{\delta_0}\right)^{1/2} u^{1-\frac{\varepsilon}{2}} - 1\right] - \rho - \mu\right\}, \tag{37}$$

$$g = \frac{1}{\sigma}\left[(\varepsilon - 1)\left(\frac{\delta_0}{\beta}\right)^{1/2} u^{\varepsilon/2} - \rho - 1\right]. \tag{38}$$

Provided that $\sigma > 1$, we see that if $\varepsilon > 2$, then the graph of (37) has a negative slope in (u, g) space, while that of (38) has a positive slope. Thus in this case the balanced growth path is uniquely given and a rise in μ depresses the balanced-growth rate, g. If $1 < \varepsilon < 2$, then both graphs are positively sloped so that there may exist multiple balanced-growth paths. Consequently, the qualitative results in the case that $0 < \phi_1 = \phi_2 = \phi < 1$ are close to those obtained in Cases (ii) and (iii).

References

Aznar-Màrquez, J. and Ruiz-Tamarit, R. (2004), "Endogenous Growth, Capital Utilization and Depreciation", *FEDRA Discussion Paper* No.24.

Chen, S-H., and Guo, J-T. (2008a), "Velocity of Money, Equilibrium (In) determinacy and Endogenous Growth", *Journal of Macroeconomics* 30, 1085-1096.

Chen, S-H., and Guo, J-T. (2008b), "On the Growth and Velocity Effects of Money", *Economics Bulletin* 5, Issue 13, 1-7.

Guo, J-T. and Lansing, K. (2007), "Maintenance Expenditures and Indeterminacy under Increasing Returns to Scale", *International Journal of Economic Theory* 3, 147-158.

Jha, S., Yip, C. and Wang, P. (2002), "Dynamics in a Transactions-Based Monetary Growth Model," *Journal of Economic Dynamics and Control* 26, 611-635.

Licandro, O. and L. A. Puch (2000), "Capital Utilization, Maintenance Costs and the Business Cycle," *Annales d'Economie et de Statistique* 58, 143-164

Licandro, O., Puch, L.A. amd Ruiz-Tamarit, R. (2001), "Optimal Growth under Endogenous Depreciation, Capital Utilization and Maintenance Costs", *Investigaciones Economicas* 25, 543-559.

Li, K.F. and Yip, C-K., (2004), "Monetary Policy and Equilibrium Indeterminacy in a Cash-in-Advance Economy with Investment", *Economics Bulletin* 5, Issue 2, 1-7.

McGrattan, E. and Schmitz Jr., J.A. (1999), "Maintenance and Repair: Too Big to Ignore," *Quarterly Review of the Federal Reserve Bank of Minneapolis* 23, No. 4, 2-23.

Mullen, K.K. and Williams, M. (2004), "Maintenance and Repair Expenditures: Determinants and Trade-offs with New Capital Goods ", *Journal of Economics and Business* 56, 483-499.

Suen, M-H. and Yip, C-K. (2005), "Superneutrality, Indeterminacy and Endogenous Growth", *Journal of Macroeconomics* 27, 579-595