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The theory of the firm under multiple uncertainties

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Abstract

Without imposing restrictions on the utility function and the probability distributions, we show the impact of multiple uncertainty (and each single uncertainty) and change in risk aversion on each input demand. In so doing, we emphasize the importance of the relationship between the inputs in this impact. Moreover, the paper provides technical contributions.

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1. Introduction

In the absence of hedging, the vast majority of the studies in uncertainty considered a single source of uncertainty such as price uncertainty or cost uncertainty. In contrast, theoretical studies in multiple uncertainty are scarce. Viaene and Zilcha (1998) considered multiple uncertainty; however they employed a single-input production function and since, as we will show in this paper, the decision analysis is sensitive to the relationship between the inputs, their results are limited. Chambers and Quiggin (2003, 2001) and Dalal and Alghalith (2009) investigated price and output uncertainty; however, they did not analyze the input demand. Therefore the relationship between the inputs was irrelevant to their models. Moreover, they employed restrictive assumptions. Even with single uncertainty, none of the previous studies, including Batra and Ullah (1974) and Pope (1980), showed the impact of uncertainty and change in risk aversion on input demand; and thus they did not show the role of the relationship between the inputs in this impact.

Other studies relied on the assumption of supermodularity in deriving comparative statics results. Examples include Athey (2002) and Milgrom and Shannon (1994), among others. However, these results require the assumption of decreasing absolute risk aversion (DARA). In addition, they mainly dealt with a single source of uncertainty. Similarly, Gollier and Pratt (1996) relied on the assumption of DARA.

In this paper, using a general framework (no restrictions on the utility functions and probability distributions), we provide comparative statics under multiple uncertainty. In so doing, we show the impact of multiple uncertainty (and each single uncertainty) and change in risk aversion on each input demand. We also show the impact of uncertainty on the optimal inputs ratio and the input productivity. In doing so, we highlight the importance of the relationship between the inputs (whether substitutes, complements, or independent) in this impact. We show that the relationship between the inputs are important both in production and in utility.

2. The model

The profit function is given by

$$\tilde{\pi} = \tilde{p} f\left(x_1, x_2\right) - \tilde{w}_1 x_1 - w_2 x_2,$$

where $\tilde{p}$ is the output price with mean $\bar{p}$, f is a neoclassical production function, x_1 is the risky input, $\tilde{w}_1$ is its price with mean $\bar{w}_1$, x_2 is the non-risky input with price w_2.[1] The risk averse firm maximizes the expected utility of the profit

$$\underset{x_1, x_2}{Max} Eu\left(\tilde{\pi}\right),$$

[1] The results hold if both inputs are risky and/or the output is random (the proofs are similar); for brevity, we focus on output price uncertainty and one risky input.

where u is a Neumann-Morgenstern utility function. The first-order conditions are

$$(\bar{p}f_1 - \bar{w}_1)\, Eu'(\tilde{\pi}^*) + f_1 Cov\,(u'(\tilde{\pi}^*), \tilde{p}) - Cov\,(u'(\tilde{\pi}^*), \tilde{w}_1) = 0, \tag{1}$$

$$(\bar{p}f_2 - w_2)\, Eu'(\tilde{\pi}^*) + f_2 Cov\,(u'(\tilde{\pi}^*), \tilde{p}) = 0. \tag{2}$$

The second-order conditions of the maximization problem are

$$H_{11} = Eu''(\tilde{\pi}^*)\,(\tilde{p}f_1 - \tilde{w}_1)^2 + f_{11}Eu'(\tilde{\pi}^*)\,\tilde{p} < 0,$$

$$H_{22} = Eu''(\tilde{\pi}^*)\,(\tilde{p}f_2 - w_2)^2 + f_{22}Eu'(\tilde{\pi}^*)\,\tilde{p} < 0,$$

and

$$|H| = \begin{vmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{vmatrix} > 0,$$

where

$$H_{12} = Eu''(\tilde{\pi}^*)\,(\tilde{p}f_1 - \tilde{w}_1)(\tilde{p}f_2 - w_2) + f_{12}Eu'(\tilde{\pi}^*)\,\tilde{p}.$$

3. The impact of the uncertainty

In this section we will establish the impact of the cost/output price risk or both risks on the optimal level of each input. That is, we will compare these values in the presence of both risks to their corresponding values under certainty, output price uncertainty, and cost uncertainty.

Proposition 1. *The introduction of output price and cost uncertainty reduces the optimal level of each input if* $f_{12} \geq 0$ and $\tilde{p}$ *and* $\tilde{w}_1$ *are statistically independent.*

Proof. Let $\bar{x}$ denote the optimal level of the input under certainty (in the absence of both risks). From (1)

$$\bar{p}f_1\left(x_{1,}^* x_2^*\right) - \bar{w}_1 > 0,$$

since $Cov\,(u'(\tilde{\pi}^*), \tilde{p}) < 0$ and $Cov\,(u'(\tilde{\pi}^*), \tilde{w}_1) > 0$. But from the necessary condition for profit maximization under certainty

$$\bar{p}f_1(\bar{x}_{1,}\bar{x}_2) - \bar{w}_1 = 0.$$

Hence, $f_1\left(x_{1,}^* x_2^*\right) > f_1(\bar{x}_{1,}\bar{x}_2)$. Similarly, using (2), we can establish $f_2\left(x_{1,}^* x_2^*\right) > f_2(\bar{x}_{1,}\bar{x}_2)$. Totally differentiating f_1 and f_2, we obtain

$$\left\{ \begin{matrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{matrix} \right\} \left\{ \begin{matrix} dx_1 \\ dx_2 \end{matrix} \right\} = \left\{ \begin{matrix} df_1 \\ df_2 \end{matrix} \right\},$$

and thus

$$dx_1 = \frac{f_{22}df_1 - f_{12}df_2}{f_{11}f_{22} - f_{12}^2},\, dx_2 = \frac{f_{11}df_2 - f_{12}df_1}{f_{11}f_{22} - f_{12}^2}.$$

Since $df_1 > 0$ and $df_2 > 0$ in response to the risks, $dx_1 < 0$ and $dx_2 < 0$ if $f_{12} \geq 0$.■

The result is intuitively appealing since the risky input should fall (due to risk aversion) in response to multiple uncertainty, while the non-risky input falls in response to output price uncertainty (this can be easily clarified by a single-input production function). However, the inputs being non-complements in production ($f_{12} \geq 0$) will guarantee that each input will not increase when the other input falls in response to the uncertainty. Thus the change in each input is the net result of two effects: the uncertainty and the technological relationship between the two inputs (substitutes, complements, or independent). The two effects will be in the same direction if the two inputs are substitutes and hence the net effect is a decrease in each input demand.

It is worth noting that the importance of the relationship between the inputs was not captured by Viaene and Zilcha's model since they employed a single-input production function. Moreover, contrary to the case of output price uncertainty or multiplicative output uncertainty where the optimal output falls regardless of the technological relationship between the inputs, we implied that the impact of multiple uncertainty on the optimal output is indeterminate if $f_{12} < 0$.

Proposition 2. *The uncertainty reduces the optimal input ratio,* x_1/x_2, *if the production function is homothetic and* $\tilde{p}$ and $\tilde{w}_1$ *are statistically independent.*

Proof. With certainty $f_1/f_2 = w_1/w_2$, but with uncertainty

$$\frac{f_1}{f_2} = \frac{\bar{w}_1}{w_2} + \frac{Cov\left(u'\left(\tilde{\pi}^*\right), \tilde{w}_1\right)}{w_2 Eu'\left(\tilde{\pi}^*\right)},$$

thus $d\left(f_1/f_2\right) > 0$ when uncertainty is added. For a homothetic production function, $f_1/f_2 = g\left(x_2/x_1\right)$ where g is a monotonic function. Thus $d\left(x_1/x_2\right) < 0$ in response to the uncertainty.■

Proposition 3. *The average productivity of the risky input increases in response to the uncertainty if the production function is homogeneous and* $\tilde{p}$ and $\tilde{w}_1$ *are independent.*

Proof. If the production function is homogeneous of degree r, then by Euler's Theorem the average productivity of x_1 can be written as

$$\frac{f}{x_1} = \frac{1}{r}\left(f_1 + f_2\frac{x_2}{x_1}\right). \tag{3}$$

We established that the introduction of uncertainty increases f_1, f_2 and x_2/x_1; thus f/x_1 increases.■

This result is also intuitive since the increase in marginal productivity increases the average productivity. It is worth noting that the previous studies did not investigate the impact of multiple uncertainty on productivity.

Proposition 4. *Starting with cost uncertainty, adding output price uncertainty reduces the optimal level of each input if* $f_{12} \geq 0$ and $\tilde{p}$ *and* $\tilde{w}_1$ *are statistically independent.*

Proof. Rewrite (1) as

$$Eu'\left(\tilde{\pi}^*\right)\left(\bar{p}f_1 - \tilde{w}_1\right) + f_1 Cov\left(u'\left(\tilde{\pi}^*\right), \tilde{p}\right) = 0.$$

Since $Cov\left(u'\left(\tilde{\pi}^*\right), \tilde{p}\right) < 0$, the equation above implies that

$$Eu'\left(\tilde{\pi}^*\right)\left(\bar{p}f_1 - \tilde{w}_1\right) > 0.$$

Define the sets A and $\sim A$ such that

$$A = \{w_1 | \bar{p}f_1 - w_1 \geq 0\},$$

$$\sim A = \{w_1 | \bar{p}f_1 - w_1 \leq 0\}.$$

For any $w_1 \in A$ and $w_1' \in\sim A$,

$$pf - w_1 x_1^* - w_2 x_2^* \geq pf - w_1' x_1^* - w_2 x_2^*;\ w_1 \in A,\ w_1' \in\sim A.$$

Since $u'' < 0$, the inequality above implies

$$u'\left(\tilde{\pi}^*\left(w_1\right)\right) \leq u'\left(\tilde{\pi}^*\left(w_1'\right)\right);\ w_1 \in A,\ w_1' \in\sim A.$$

Therefore,

$$S = \max_{w_1 \in A} u'\left(\tilde{\pi}^*\right) \leq I = \min_{w_1 \in \sim A} u'\left(\tilde{\pi}^*\right).$$

Thus,

$$\frac{S}{u'\left(E_p \tilde{\pi}^*\right)} \leq \frac{I}{u'\left(E_p \tilde{\pi}^*\right)},$$

where E_p denotes the expectation with respect to $\tilde{p}$ for a given w_1. Since S and I are both positive, there must exist a positive constant t such that

$$\frac{E_p S}{u'\left(E_p \tilde{\pi}^*\right)} \leq t \leq \frac{E_p I}{u'\left(E_p \tilde{\pi}^*\right)},$$

so that

$$tu'\left(E_p \tilde{\pi}^*\right) \geq E_p S \geq E_p u'\left(\tilde{\pi}^*\right),\ w_1 \in A, \tag{4}$$

where the last inequality in (4) holds since S is a maximum. Now (4) implies

$$\left(\bar{p} f_1 - w_1\right) tu'\left(E_p \tilde{\pi}^*\right) \geq \left(\bar{p} f_1 - w_1\right) E_p u'\left(\tilde{\pi}^*\right),\ w_1 \in A, \tag{5}$$

since $\bar{p} f_1 - w_1 > 0$ for $w_1 \in A$. Similarly,

$$\left(\bar{p} f_1 - w_1\right) tu'\left(E_p \tilde{\pi}^*\right) \geq \left(\bar{p} f_1 - w_1\right) E_p u'\left(\tilde{\pi}^*\right),\ w_1 \in\sim A, \tag{6}$$

since $\left(\bar{p} f_1 - w_1\right) < 0$ for $w_1 \in\sim A$. Thus the inequalities in (5) and (6) hold for all values of w_1, and taking expectations with respect to $\tilde{w}_1$, we obtain

$$tE_{w_1}\left(\bar{p} f_1 - \tilde{w}_1\right) u'\left(E_p \tilde{\pi}^*\right) \geq E\left(\bar{p} f_1 - \tilde{w}_1\right) u'\left(\tilde{\pi}^*\right) > 0. \tag{7}$$

Since $t > 0$, (7) implies

$$E_{w_1}\left(\bar{p} f_1 - \tilde{w}_1\right) u'\left(E_p \tilde{\pi}^*\right) > 0. \tag{8}$$

Now, define $a_1 \equiv E_{w_1}\left(\bar{p} f_1 - \tilde{w}_1\right) u'\left(E_p \tilde{\pi}\right)$ and thus $da_1 > 0$ in response to the introduction of the output price risk. Similarly, define $a_2 \equiv E_{w_1}\left(\bar{p} f_2 - w_2\right) u'\left(E_p \tilde{\pi}\right)$ and thus $da_2 > 0$ in response to the output price risk. Totally differentiating a_1 and a_2 (holding the parameters constant), we obtain

$$\left\{\begin{array}{cc} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{12} & \bar{H}_{22} \end{array}\right\} \left\{\begin{array}{c} dx_1 \\ dx_2 \end{array}\right\} = \left\{\begin{array}{c} da_1 \\ da_2 \end{array}\right\},$$

where $\bar{H}$ is the Hessian in the absence of the output price risk and $\bar{H}_{12} = f_{12} \bar{p} E_{w_1} u'\left(E_p \tilde{\pi}\right)$; thus

$$dx_1 = \frac{\bar{H}_{22} da_1 - \bar{H}_{12} da_2}{\bar{H}_{11} \bar{H}_{22} - \bar{H}_{12}^2},\ dx_2 = \frac{\bar{H}_{11} da_2 - \bar{H}_{12} da_1}{\bar{H}_{11} \bar{H}_{22} - \bar{H}_{12}^2}.$$

Therefore $dx_1 < 0$ and $dx_2 < 0$.■

It is worth noting that this result is stronger than the standard results on aversion to one risk in the presence of another (see Golllier (2001) and Pratt (1988)). In addition, the proof provides technical contributions.

Proposition 5. *Given the statistical independence between $\tilde{p}$ and $\tilde{w}_1$, starting with output price uncertainty, adding cost uncertainty (i) reduces the optimal level of the risky input (ii) reduces the optimal level of the non-risky input if the inputs are substitutes (in preferences).*

Proof. Define $b_1 \equiv E_p u' \left(E_{w_1}\tilde{\pi}\right)\left(\tilde{p}f_1 - \bar{w}_1\right)$ and $b_2 \equiv E_p u' \left(E_{w_1}\tilde{\pi}\right)\left(\tilde{p}f_2 - w_2\right)$; we can show $db_1 > 0$ and $db_2 = 0$ in response to the cost uncertainty (the proof is similar to the proof of Proposition 4). Totally differentiating b_1 and b_2 (holding the parameters constant), we obtain

$$\left\{\begin{array}{cc} \dddot{H}_{11} & \dddot{H}_{12} \\ \dddot{H}_{12} & \dddot{H}_{22} \end{array}\right\}\left\{\begin{array}{c} dx_1 \\ dx_2 \end{array}\right\} = \left\{\begin{array}{c} db_1 \\ 0 \end{array}\right\},$$

where $\dddot{H}$ is the Hessian in the absence of cost risk, thus

$$dx_1 = \frac{\dddot{H}_{22}db_1}{\left|\dddot{H}\right|}, dx_2 = \frac{-\dddot{H}_{12}db_1}{\left|\dddot{H}\right|}.$$

Hence, $dx_1 < 0$; when the inputs are substitutes $Eu_{12} \equiv H_{12} > 0$; thus, $dx_2 < 0$. Also, $dx_2 \geq 0$ if the inputs are complements (or independent), since $Eu_{12} \leq 0$.■

The result is also intuitive since the risky input must fall in response to cost uncertainty, whereas the non-risky input reacts according to the preferences relationship between the two inputs. Thus the non-risky input decreases (increases) if the inputs are substitutes (complements).

4. The impact of change in risk aversion

In this section we show the impact of an increase in risk aversion on the optimal level of each input.

Proposition 6. *Given the statistical independence between $\tilde{p}$ and $\tilde{w}_1$, when risk aversion increases the optimal level of each input falls if the inputs are substitutes or independent.*

Proof. For each fixed value of $\tilde{p}$, define $\hat{w}_1$ by $pf_1\left(\mathbf{x}^*\right) - \hat{w}_1 = 0$ and let $\hat{\pi}$ be the profit when $w_1 = \hat{w}_1$. Also let $\mathbf{x}^*$ be the optimal input vector for firm1, respectively. Assume that firm 1 is more risk averse than firm 2. The first-order condition for firm 1 can be written as

$$E_{w_1}\frac{u_1'\left(\tilde{\pi}^*\right)}{u_1'\left(\hat{\pi}\right)}\left(pf_1\left(\mathbf{x}^*\right) - \tilde{w}_1\right) = 0, \tag{9}$$

where u_1 is firm 1's utility function and $u_1'\left(\hat{\pi}\right)$ is a constant. Equation (9) can be rewritten as

$$\int_{\tilde{w}_1<\hat{w}_1}\frac{u_1'\left(\tilde{\pi}^*\right)}{u_1'\left(\hat{\pi}\right)}\left(pf_1 - \tilde{w}_1\right)\Gamma\left(\tilde{w}_1\right)d\tilde{w}_1 + \int_{\tilde{w}_1>\hat{w}_1}\frac{u_1'\left(\tilde{\pi}^*\right)}{u_1'\left(\hat{\pi}\right)}\left(pf_1 - \tilde{w}_1\right)\Gamma\left(\tilde{w}_1\right)d\tilde{w}_1 = 0, \tag{10}$$

where $f_1 = f_1\left(\mathbf{x}^*\right)$. The corresponding expression for firm 2 is

$$\int_{\tilde{w}_1<\hat{w}_1}\frac{u_2'\left(\tilde{\pi}^*\right)}{u_2'\left(\hat{\pi}\right)}\left(pf_1 - \tilde{w}_1\right)\Gamma\left(\tilde{w}_1\right)d\tilde{w}_1 + \int_{\tilde{w}_1>\hat{w}_1}\frac{u_2'\left(\tilde{\pi}^*\right)}{u_2'\left(\hat{\pi}\right)}\left(pf_1 - \tilde{w}_1\right)\Gamma\left(\tilde{w}_1\right)d\tilde{w}_1. \tag{11}$$

Subtracting (10) from (11) yields

$$\int_{\tilde{w}_1<\hat{w}_1}\left[\frac{u_2'(\tilde{\pi}^*)}{u_2'(\hat{\pi})}-\frac{u_1'(\tilde{\pi}^*)}{u_1'(\hat{\pi})}\right](pf_1-\tilde{w}_1)\,\Gamma d\tilde{w}_1+\int_{\tilde{w}_1>\hat{w}_1}\left[\frac{u_2'(\tilde{\pi}^*)}{u_2'(\hat{\pi})}-\frac{u_1'(\tilde{\pi}^*)}{u_1'(\hat{\pi})}\right](pf_1-\tilde{w}_1)\,\Gamma d\tilde{w}_1. \quad (12)$$

An increase in $\tilde{w}_1$ decreases $\tilde{\pi}^*$. In the first integral $(pf_1(\mathbf{x}^*)-\tilde{w}_1)$ is positive, Pratt showed that the term in square brackets is also positive. These inequalities are both reversed in the second integral. Consequently (12) is positive and thus (11) is positive. By the independence assumption,

$$E\frac{u_2'(\tilde{\pi}^*)}{u_2'(\hat{\pi})}(\tilde{p}f_1-\tilde{w}_1)=E_pE_{w_1}\frac{u_2'(\tilde{\pi}^*)}{u_2'(\hat{\pi})}(\tilde{p}f_1-\tilde{w}_1)>0.$$

Define $\alpha_1\equiv Eu_2'(\tilde{\pi})(\tilde{p}f_1(\mathbf{x})-\tilde{w}_1)$; thus $d\alpha_1>0$ when risk aversion increases. Similarly, we can show that $\alpha_2\equiv Eu_2'(\tilde{\pi})(\tilde{p}f_2(\mathbf{x})-w_2)>0$ when risk aversion increases. Totally differentiating α_1 and α_2 (holding the parameters constant), we obtain

$$d\alpha_1=H_{11}dx_1+H_{12}dx_2>0, \quad (13)$$

$$d\alpha_2=H_{22}dx_2+H_{12}dx_1>0. \quad (14)$$

From (13) and (14)

$$\left\{\begin{array}{cc}H_{11} & H_{12}\\ H_{12} & H_{22}\end{array}\right\}\left\{\begin{array}{c}dx_1\\ dx_2\end{array}\right\}=\left\{\begin{array}{c}d\alpha_1\\ d\alpha_2\end{array}\right\}.$$

Thus

$$dx_1=\frac{H_{22}da_1-H_{12}da_2}{|H|},dx_2=\frac{H_{11}da_2-H_{12}da_1}{|H|},$$

and thus $dx_1<0$ and $dx_2<0$ when risk aversion increases.■

5. Conclusion

This paper highlights the importance of the relationship between the inputs in determining the impact of multiple uncertainty on the input demand. In the presence of multiple uncertainty, each input demand is less than its certainty-equivalent level, given the two inputs are substitutes in production. This is due to two changes. First, the fact that the risky input falls in response to multiple uncertainty, while the non-risky input falls in response to output price uncertainty. Second, the inputs being substitutes causes each input to decrease when the other input falls due to uncertainty. The two changes will be in the same direction if the two inputs are substitutes and hence the net effect is a decrease in each input demand. Consequently, in the presence of multiple uncertainty, each input demand is less than the input demand in the presence of only cost uncertainty, given the two inputs are substitutes in production/preferences. Also, in the presence of multiple uncertainty, the demand for each input is less than its level in the presence of only output price uncertainty, given the inputs are substitutes in preferences. Moreover, when risk aversion increases, both inputs fall if the inputs are substitutes in preferences.

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