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A note on the existence of monetary equilibrium in a stochastic OLG model with a finite state space

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Abstract

The present paper provides a simple and complete proof of the existence of a stationary monetary equilibrium for a stochastic overlapping generations model with a finite state space. Differently from previous studies, we show that all the prices are positive without the Frobenius theorem.

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1. Introduction

It is well-known that the class of overlapping generations (henceforth, OLG) models is a useful class of models for exploring properties of equilibrium over time. The aim of this paper is to provide a proof of the existence of a stationary monetary equilibrium (henceforth, SME) for a stochastic OLG model with a finite state space, which was studied by Magill and Quinzii (2003, henceforth, MQ). MQ, of course, showed the existence of a SME. They, however, referred to the paper by Gottardi (1996) for the fixed point argument and did not provide a complete proof of the existence of a SME. Since Gottardi dealt with a more complicated OLG model with heterogenous agents and several securities, it is worthwhile to provide a simple and complete proof of the existence that is tailored to the simple, but rather canonical, model of monetary OLG economy studied by MQ.

This paper shows the existence of a SME by applying the Brouwer fixed-point theorem.[1] While Gottardi and MQ applied the Frobenius theorem to show that all the prices are positive, this paper shows it without the Frobenius theorem.[2] Moreover, this paper demonstrates that we can find a SME for a broader range of OLG models than MQ's.

2. The Model

2.1. Fundamentals

We consider the same model as that of Magill and Quinzii (2003), that is, a one-good, pure-exhange overlapping generations economy in which agents live for two periods and have random endowments. The uncertainty is modeled by the realization of one of finite number of the shocks at the beginning of each period, $s \in \boldsymbol{S} = \{1, \ldots, S\}$, s_0 being the initial shock. Let $\boldsymbol{s}_t = (s_0, \ldots, s_t) \in \boldsymbol{S}^{1+t}$ denote the history of the shocks occurred from period 0 to period t and let $\Sigma_t := \boldsymbol{S}^{1+t}$ denote the set of all such histories up to date t. Note that $\Sigma_0 = \boldsymbol{S}$ and $\Sigma_\infty = \boldsymbol{S}^\infty$. Also let $\Sigma := \cup_{t=0}^{\infty}\Sigma_t$ denote the collection of all such histories for all periods. $\boldsymbol{s} = (s_0, \ldots, s_t, \ldots) \in \Sigma$ is denoting a typical path of the event-tree Σ. We assume that shocks follow a first-order Markov process and denote by P the induced probability on the event-tree Σ.

In each period $t \geq 0$, a new generation enter the economy after the resolution of the period t uncertainty. Therefore, it is impossible for the young and the old to enter into risk sharing contracts. We assume that there is only one agent per generation and that all agents born in and after the initial period, called "newly born agents," share the same endowment and preference structures. At date-event $\boldsymbol{s}_t \in \Sigma$, a newly born agent is endowed with the random endowment stream $\omega(\boldsymbol{s}_t) = (\omega^1(\boldsymbol{s}_t), (\omega^2(\boldsymbol{s}_t, s'))_{s' \in \boldsymbol{S}}) \in \mathbf{R}_+ \times \mathbf{R}_+^S$. We assume that the endowment stream depends only on the shock s_t realized when the agent is young, i.e, $\omega(\boldsymbol{s}_t) \equiv \omega(s_t) = (\omega^1_{s_t}, (\omega^2_{s_t s'})_{s' \in \boldsymbol{S}})$ for $\boldsymbol{s}_t = (s_0, \ldots, s_t)$.

All newly born agents maximize the expected utility of their lifetime consumption streams, with the same utility indices. A newly born agent born at date-event $\boldsymbol{s}_t \in \Sigma$ ranks the possible consumption streams $\mathbf{c}_t(\boldsymbol{s}_t) = (c^1(\boldsymbol{s}_t), (c^2(\boldsymbol{s}_t, s'))_{s' \in \boldsymbol{S}}) \in \mathbf{R}_+ \times \mathbf{R}_+^S$ according to a utility function $U_{\boldsymbol{s}_t} : \mathbf{R}_+^{1+S} \to \mathbf{R}$.

To close the model, we introduce the old generation present in the initial period, which we call the "initial old." This generation contains one old agent, whom is endowed with

[1] See also Manuelli (1990). He showed the existence of a stationary monetary equilibrium for a stochastic OLG model with a continuous state space by applying the Shauder fixed-point theorem.

[2] See Debreu and Herstein (1953) and Takayama (1974) for more details on the Frobenius theorem.

$\omega^0(\boldsymbol{s}_0)$ units of the consumption good at date-event $\boldsymbol{s}_0$. At date-event $\boldsymbol{s}_0 \in \Sigma_0$, he/she ranks the possible consumption plan $c_0^2(\boldsymbol{s}_0) \in \mathbf{R}_+$ according to a utility function $U^0_{\boldsymbol{s}_0} : \mathbf{R}_+ \to \mathbf{R}$.

2.2. Assumptions on Fundamentals

Following Magill and Quinzii (2003), we impose the following system of assumptions on the model.

Assumption 1 (Markov Structure). *There exists a Markov transition matrix* $\rho = [\rho_{ss'}]_{s,s'\in\boldsymbol{S}}$ *with* $\rho_{ss'} > 0$ *for all* $s, s' \in \boldsymbol{S}$ *such that* $P(s_{t+1} = s'|s_t = s) = \rho_{ss'}$ *for all* $s, s' \in \boldsymbol{S}$.

Assumption 2 (Positive Endowment).

1. $\omega(s) = (\omega^1_s, (\omega^2_{ss'})_{s'\in\boldsymbol{S}}) \in \mathbf{R}^{1+S}_{++}$ *for all* $s \in \boldsymbol{S}$;
2. $\omega^2_0(s_0) \in \mathbf{R}_{++}$ *for all* $s_0 \in \boldsymbol{S}$.

Assumption 3 (Preferences).

1. *There exist increasing, concave, and differentiable functions* $u_1, u_2 : \mathbf{R}_+ \to \mathbf{R}$ *with* $\lim_{c\downarrow 0} u_i'(c) = \infty$ *for* $i = 1, 2$ *such that*
$$U_{\boldsymbol{s}_t}(c(\boldsymbol{s}_t)) = u_1(c^1(\boldsymbol{s}_t)) + \sum_{s'\in\boldsymbol{S}} u_2(c^2(\boldsymbol{s}_t, s'))\rho_{ss'}$$
for all $\boldsymbol{s}_t \in \Sigma$ *and all* $c(\boldsymbol{s}_t) = (c^1(\boldsymbol{s}_t), (c^2(\boldsymbol{s}_t, s'))_{s'\in\boldsymbol{S}}) \in \mathbf{R}_+ \times \mathbf{R}^S_+$;
2. $-cu_2''(c)/u_2'(c) \leq 1$ *for all* $c \in \mathbf{R}_{++}$;
3. $U^0_{\boldsymbol{s}_0}(c^2_0(\boldsymbol{s}_0)) = c^2_0(\boldsymbol{s}_0)$.

2.3. Equilibrium Price Processes (Vectors)

To define equilibria as consequences of intergenerational trade, we assume that there is an infinitely-lived (outside) asset available in positive supply, normalized to 1, which yields no dividends (usually called fiat money). The asset is initially held by the initial old and is then exchanged (unless prices are zero) at each date between the old and the young. Let $q(\boldsymbol{s}_t) \in \mathbf{R}_+$ denote the price of the asset at date-event $\boldsymbol{s}_t \in \Sigma$. A newly born agent at date-event $\boldsymbol{s}_t$ chooses $z(\boldsymbol{s}_t) \in \mathbf{R}$ to maximize $U_{\boldsymbol{s}_t}(c_t(\boldsymbol{s}_t))$ subject to budget constraints

$$\begin{aligned} c^1_t(\boldsymbol{s}_t) &= \omega^1(\boldsymbol{s}_t) - q(\boldsymbol{s}_t)z(\boldsymbol{s}_t), \\ (\forall s' \in \boldsymbol{S}) \;\; c^2_{t+1}(\boldsymbol{s}_t) &= \omega^2(\boldsymbol{s}_t, s') + q(\boldsymbol{s}_t, s')z(\boldsymbol{s}_t). \end{aligned}$$

Under Assumption 3.1, the necessary and sufficient condition for this choice problem is provided by

$$(\forall \boldsymbol{s}_t \in \Sigma) \quad q(\boldsymbol{s}_t)u_1'(\omega^1_{s_t} - q(\boldsymbol{s}_t)z(\boldsymbol{s}_t)) = \sum_{s'\in\boldsymbol{S}} q(\boldsymbol{s}_t, s')u_2'(\omega^2_{s_t s'} + q(\boldsymbol{s}_t, s')z(\boldsymbol{s}_t))\rho_{s_t s'}$$

Since, in any equilibrium, agents choice must be consistent with the market clearing condition, $z(\boldsymbol{s}_t) = 1$ for all $\boldsymbol{s}_t \in \Sigma$, we are then led to the definition of an equilibrium.

Definition 1. $(q(\boldsymbol{s}_t))_{\boldsymbol{s}_t \in \Sigma}$ with $q(\boldsymbol{s}_t) \in \mathbf{R}_+$ for all $\boldsymbol{s}_t \in \Sigma$ is an *equilibrium price process* if

$$(\forall \boldsymbol{s}_t \in \Sigma) \quad q(\boldsymbol{s}_t) u_1'(\omega^1_{s_t} - q(\boldsymbol{s}_t)) = \sum_{s' \in \boldsymbol{S}} q(\boldsymbol{s}_t, s') u_2'(\omega^2_{s_t s'} + q(\boldsymbol{s}_t, s')) \rho_{s_t s'}. \qquad (\mathcal{E})$$

We concentrate our attention on "stationary equilibrium" rather than equilibrium processes. The next definition defines stationary equilibrium price vectors:

Definition 2. $(q^*_s)_{s \in \boldsymbol{S}} \in \mathbf{R}^S_+$ is a (*strongly*) *stationary equilibrium price vector* if

$$(\forall s \in \boldsymbol{S}) \quad q^*_s u_1'(\omega^1_s - q^*_s) = \sum_{s' \in \boldsymbol{S}} q^*_{s'} u_2'(\omega^2_{ss'} + q^*_{s'}) \rho_{ss'}. \qquad (\mathcal{E}^*)$$

3. Results

One can easily find a trivial stationary equilibrium price vector.

Proposition 0. $\bar{\boldsymbol{q}} \equiv \mathbf{0}$ *is a stationary price vector.*

Proof. $\bar{\boldsymbol{q}} \equiv \mathbf{0}$, i.e., $\bar{q}_s = 0$ for all $s \in \boldsymbol{S}$, obviously satisfies Eq.($\mathcal{E}^*$), so that it is a stationary equilibrium price vector. Q.E.D.

We add an assumption to present a MQ's result. Define the $S \times S$ matrix Π^0 by

$$\Pi^0 = [\pi^0_{ss'}]_{s,s' \in \boldsymbol{S}} := \left[\frac{u_2'(\omega^2_{ss'}) \rho_{ss'}}{u_1'(\omega^1_s)} \right]_{s,s' \in \boldsymbol{S}}.$$

Since, under Assumption 2, Π^0 is a matrix with positive coefficients, it follows from the Frobenius theorem that Π^0 has a unique positive eigenvalue associated with a positive eigenvector. Let $\lambda_f(\Pi^0)$ denote this eigenvalue.

Assumption 4. $\lambda_f(\Pi_0) > 1$.

Proposition 1 (Magill and Quinzii, 2003). *Under Assumptions 1,2,3, and 4, a unique positive stationary equilibrium price vector exists.*

Proof. See the proof of Proposition 1 in Magill and Quinzii (2003). One can find that the proof of the result that all the prices are positive crucially depends on the Frobenius theorem. Q.E.D.

Notice that in the proof by Magill and Quinzii (2003), Assumption 2, i.e., the hypothesis that $\omega^2 \gg 0$, plays an important role to apply the Frobenius theorem. However, if $\omega^2_{ss'} = 0$ for some pair of s and s', then Π^0 is no longer well-defined, so that we can no longer ensure that all the prices are positive by the Frobenius theorem. Here, we provide an existence result that includes such cases.

We now describe our main result. Instead of Assumptions 2 and 4 of Magill and Quinzii (2003), we impose the following assumptions.

Assumption 2′. $\omega_s^1 \in \mathbf{R}_{++}$ *for all* $s \in \boldsymbol{S}$.

Assumption 4′. $\omega_{tt'}^2 = 0$ *for some* $t, t' \in \boldsymbol{S}$ *or* $\min_{s\in\boldsymbol{S}} \sum_{s'\in\boldsymbol{S}}[u_2'(\omega_{ss'}^2)/u_1'(\omega_{s'}^1)]\rho_{ss'} > 1$.

Note that the model has presumed that $\omega_{ss'}^2$ is nonnegative for all $s, s' \in \boldsymbol{S}$. Hence, $\omega_{ss'}^2$ must be positive for all $s, s' \in \boldsymbol{S}$ and, under Assumption 4′, it must follow that $\min_{s\in\boldsymbol{S}} \sum_{s'\in\boldsymbol{S}}[u_2'(\omega_{ss'}^2)/u_1'(\omega_{s'}^1)]\rho_{ss'} > 1$ whenever it is false that $\omega_{tt'}^2 = 0$ for some $t, t' \in \boldsymbol{S}$.

We then show that a nontrivial stationary equilibrium price vector exists and it is positive without the Frobenius theorem.

Proposition 1′. *Under Assumptions 1,2′,3, and 4′, a unique positive stationary equilibrium price vector exists.*

Proof. (Existence) Our proof strategy is constructed from four steps:

1. Define some continuous mapping Φ of $\mathbf{R}_{++}^S$ to $\mathbf{R}_{++}^S$;
2. Find a compact and convex set $\mathcal{A}$ such that $\mathcal{A} \subset \mathbf{R}_{++}^S$ and Φ maps $\mathcal{A}$ into itself;
3. Applying the Brouwer fixed-point theorem to Φ on $\mathcal{A}$, we show the existence of a fixed point $\boldsymbol{a}^* \in \mathcal{A}$;[3]
4. Verify that we can construct a positive stationary equilibrium price vector from $\boldsymbol{a}^*$.

1. Our first task is to define the continuous mapping Φ of $\mathbf{R}_{++}^S$ to $\mathbf{R}_{++}^S$. For each $s \in \boldsymbol{S}$, let $Q_s := [0, \omega_s^1]$ and its interior be denoted by int.Q_s. For all $s \in \boldsymbol{S}$, define the function $f_s : Q_s \times \mathbf{R}_+ \to \mathbf{R}$ by $f_s(q, a) := qu_1'(\omega_s^1 - q) - a$ for all $(q, a) \in Q_s \times \mathbf{R}_+$. Notice that, for all $s \in \boldsymbol{S}$ and all $a \in \mathbf{R}_+$, $f_s(\,\cdot\,, a)$ is continuous and increasing on int.Q_s by Assumption 3.1. Also notice that, for all $s \in \boldsymbol{S}$ and all $a \in \mathbf{R}_{++}$, $f_s(0, a) = -a < 0$ and $\lim_{q\uparrow\omega_s^1} f_s(q, a) = \infty > 0$. Hence, for all all $s \in \boldsymbol{S}$ and $a \in \mathbf{R}_{++}$, the intermediate value theorem ensures the existence and the uniqueness of $\hat{q}_s(a) \in$ int.Q_s such that $f_s(\hat{q}_s(a), a) = 0$. Moreover, since $\partial f_s/\partial q = u_1' - qu_1'' > 0$ and $\partial f_s/\partial a = -1 < 0$, the implicit function theorem implies that, for all $s \in \boldsymbol{S}$, $\hat{q}_s(\,\cdot\,)$ is a continuous and increasing function on $\mathbf{R}_{++}$.

Define the mapping $\phi : \mathbf{R}_{++}^S \to \mathbf{R}_{++}^S$ by $\phi(\boldsymbol{a}) := (\hat{q}_s(a_s))_{s\in\boldsymbol{S}}$ for all $\boldsymbol{a} = (a_s)_{s\in\boldsymbol{S}} \in \mathbf{R}_{++}^S$. Also define the mapping $\psi : \mathbf{R}_{++}^S \to \mathbf{R}_{++}^S$ by $\psi(\boldsymbol{q}) := (\sum_{s'\in\boldsymbol{S}} q_{s'}u_2'(\omega_{ss'}^2 + q_{s'})\rho_{ss'})_{s\in\boldsymbol{S}}$ for all $\boldsymbol{q} = (q_s)_{s\in\boldsymbol{S}} \in \mathbf{R}_{++}^S$. It is easy to verify that both ϕ and ψ are continuous on $\mathbf{R}_{++}^S$. Moreover, ϕ is increasing and ψ is nondecreasing on $\mathbf{R}_{++}^S$. It is obvious that ϕ is increasing, since $\hat{q}_s(\,\cdot\,)$ is increasing for all $s \in \boldsymbol{S}$. On the other hand, to see that ψ is nondecreasing, notice that, for all $s, s' \in \boldsymbol{S}$, $\partial[qu_2'(\omega_{ss'}^2 + q)]/\partial q = u_2' + qu_2'' \geq u_2' \cdot (1 + (q + \omega_{ss'}^2)u_2''/u_2') \geq 0$, where the first inequality follows from concavity of u_2 and the fact that $\omega_{ss'}^2 \geq 0$, and the second inequality follows from Assumption 3.2. Hence, $qu_2'(\omega_{ss'}^2 + q)$ is nondecreasing in q for all $s, s' \in \boldsymbol{S}$. Therefore, it follows that $\psi(\boldsymbol{q}^0) \leq \psi(\boldsymbol{q}^1)$ for all $\boldsymbol{q}^0, \boldsymbol{q}^1 \in \mathbf{R}_{++}^S$ such that $\boldsymbol{q}^0 \leq \boldsymbol{q}^1$.

Now, define the mapping $\Phi : \mathbf{R}_{++}^S \to \mathbf{R}_{++}^S$ by $\Phi(\boldsymbol{a}) := (\psi \circ \phi)(\boldsymbol{a})$ for all $\mathbf{R}_{++}^S$. It follows from continuity and monotonicity of ϕ and ψ that Φ is continuous and nondecreasing on $\mathbf{R}_{++}^S$.

2. Next, we find a compact and convex set $\mathcal{A}$ such that $\mathcal{A} \subset \mathbf{R}_{++}^S$ and Φ maps $\mathcal{A}$ into itself. Let $A_s := [0, \omega_s^1 u_1'(\omega_s^1)]$ for all $s \in \boldsymbol{S}$ and $A := \bigcap_{s\in\boldsymbol{S}} A_s$. The interiors of A_s

[3] See, for example, Takayama (1974) for details on the Brouwer fixed-point theorem.

and A are denoted by int.A_s and int.A, respectively. For all $s, s' \in \boldsymbol{S}$, define the function $z_s(s', \cdot\,) : A_{s'} \to \mathbf{R}_{++}$ by

$$(\forall a \in A_{s'}) \quad z_s(s', a) := \frac{u_2'(\omega_{ss'}^2 + a/u_1'(\omega_{s'}^1))}{u_1'(\omega_{s'}^1 - a/u_1'(\omega_{s'}^1))}.$$

Then, it follows from Assumption 3.1 that $z_s(s', \cdot\,)$ is continuous and decreasing on int.$A_{s'}$. Notice that it follows from the definition of $\hat{q}_s(\,\cdot\,)$ and concavity of u_1 that $\hat{q}_s(a) = a/u_1'(\omega_s^1 - \hat{q}_s(a)) \leq a/u_1'(\omega_s^1)$ for all $s \in \boldsymbol{S}$ and all $a \in \mathbf{R}_{++}$. Thus, it follows from concavity of u_1 and u_2 that

$$(\forall s, s' \in \boldsymbol{S})(\forall a \in \text{int}.A_{s'}) \quad z_s(s', a) \leq \frac{u_2'(\omega_{ss'}^2 + \hat{q}_{s'}(a))}{u_1'(\omega_{s'}^1 - \hat{q}_{s'}(a))}. \tag{1}$$

Define the function $\hat{z} : A \to \mathbf{R}_{++}$ by $\hat{z}(a) := \min_{s \in \boldsymbol{S}} \sum_{s' \in \boldsymbol{S}} z_s(s', a)\rho_{ss'}$ for all $a \in A$. By properties of $z_s(s', \cdot\,)$, $\hat{z}$ is continuous and decreasing on int.A. Let $\hat{z}(0) := \lim_{a \downarrow 0} \hat{z}(a)$. If $\omega_{ss'}^2 = 0$ for some $s, s' \in \boldsymbol{S}$, then it follows from Assumption 3.1 that $\hat{z}(0) = \infty > 1$. On the other hand, if $\omega_{ss'}^2 > 0$ for all $s, s' \in \boldsymbol{S}$, then Assumption 4′ ensures that $\hat{z}(0) > 1$. Hence, in all cases, we have $\hat{z}(0) > 1$. Since $\hat{z}$ is continuous and decreasing on int.A, there exists at least one $\underline{a} \in \text{int}.A$ such that $\hat{z}(0) > \hat{z}(\underline{a}) \geq 1$. Also let $\overline{a} := \overline{\omega}^1 u_2'(\underline{\omega}^2 + \underline{q})$, where $\overline{\omega}^1 := \max_{s \in \boldsymbol{S}} \omega_s^1$, $\underline{\omega}^2 := \min_{s, s' \in \boldsymbol{S}} \omega_{ss'}^2$, and $\underline{q} := \min_{s \in \boldsymbol{S}} \hat{q}_s(\underline{a}) > 0$. Notice that $\overline{\omega}^1 \geq \omega_{s'}^1$, $\underline{\omega}^2 \leq \omega_{ss'}^2$, and $\underline{q} \leq \hat{q}_{s'}(a)$ for all $s, s' \in \boldsymbol{S}$ and all $a \geq \underline{a}$. Therefore, it follows from concavity of u_2 that

$$\sum_{s' \in \boldsymbol{S}} \omega_{s'}^1 u_2'(\omega_{ss'}^2 + \hat{q}_{s'}(a_s))\rho_{ss'} \leq \overline{a} \tag{2}$$

for all $s \in \boldsymbol{S}$ and all $(a_{s'})_{s' \in \boldsymbol{S}}$ such that $a_{s'} \geq \underline{a}$ for all $s' \in \boldsymbol{S}$.

We claim that $\underline{a} < \overline{a}$. To see this, notice that

$$(\forall s' \in \boldsymbol{S}) \quad \underline{a} < \omega_{s'}^1 u_1'(\omega_{s'}^1) \leq \overline{\omega}^1 u_1'(\omega_{s'}^1) = \overline{a} \frac{u_1'(\omega_{s'}^1)}{u_2'(\underline{\omega}^2 + \underline{q})}, \tag{3}$$

where the first (strict) inequality follows from the fact that $\underline{a} \in \text{int}.A$, the second inequality follows from the fact that $\omega_{s'}^1 \leq \overline{\omega}^1$, and the last equality follows from the definition of $\overline{\omega}^1$. Then, it follows that

$$(\forall s, s' \in \boldsymbol{S}) \quad \overline{a} > \underline{a} \frac{u_2'(\underline{\omega}^2 + \underline{q})}{u_1'(\omega_{s'}^1)} \geq \underline{a} \frac{u_2'(\omega_{ss'}^2 + \hat{q}_s'(\underline{a}))}{u_1'(\omega_{s'}^1)} \geq \underline{a} \frac{u_2'(\omega_{ss'}^2 + \hat{q}_s(\underline{a}))}{u_1'(\omega_{s'}^1 - \hat{q}_s(\underline{a}))} \geq \underline{a} z_s(s', \underline{a}), \tag{4}$$

where the first (strict) inequality follows from Eq.(3), the second follows from concavity of u_2 and the facts that $\underline{\omega}^2 \leq \omega_{ss'}^2$ and $\hat{q}_s(\underline{a}) \geq \underline{q}$ for all $s, s' \in \boldsymbol{S}$, the third follows from concavity of u_1 and the fact that $\hat{q}_s(\underline{a}) > 0$, and the last inequality follows from Eq.(1). Hence we have

$$(\forall s \in \boldsymbol{S}) \quad \overline{a} = \overline{a} \sum_{s' \in \boldsymbol{S}} \rho_{ss'} > \underline{a} \sum_{s' \in \boldsymbol{S}} z_s(s', \underline{a})\rho_{ss'} \geq \underline{a}\hat{z}(\underline{a}) \geq \underline{a},$$

where the first equality follows from the definition of ρ, the second (strict) inequality follows from Eq.(4), the third follows from the definition of $\hat{z}$, and the last follows from the fact that $\hat{z}(\underline{a}) \geq 1$. This establishes the claim.

We also claim that $\underline{\boldsymbol{a}} \leq \Phi(\boldsymbol{a}) \ll \overline{\boldsymbol{a}}$ for all $\boldsymbol{a} \in \mathbf{R}^S_{++}$ with $\underline{\boldsymbol{a}} \leq \boldsymbol{a} \leq \overline{\boldsymbol{a}}$, where $\overline{\boldsymbol{a}} := (\overline{a}, \ldots, \overline{a}), \underline{\boldsymbol{a}} := (\underline{a}, \ldots, \underline{a}) \in \mathbf{R}^S_{++}$. Notice that, for all $\boldsymbol{a} \in \mathbf{R}^S_{++}$ with $\boldsymbol{a} \geq \underline{\boldsymbol{a}}$ and all $s \in \boldsymbol{S}$, $\sum_{s' \in \boldsymbol{S}} \hat{q}_{s'}(a_{s'}) u_2'(\omega^2_{ss'} + \hat{q}_{s'}(a_{s'}))\rho_{ss'} < \sum_{s' \in \boldsymbol{S}} \omega^1_{s'} u_2'(\omega^2_{ss'} + \hat{q}_{s'}(a_{s'}))\rho_{ss'} \leq \overline{a}$, where the first (strict) inequality follows from the fact that $\hat{q}_s(a) < \omega^1_s$ for all positive a, and the second inequality follows from Eq.(2). Hence, $\Phi(\boldsymbol{a}) \ll \overline{\boldsymbol{a}}$ for all $\boldsymbol{a} \in \mathbf{R}^S_{++}$ with $\boldsymbol{a} \geq \underline{\boldsymbol{a}}$. On the other hand, notice that

$$\begin{aligned}
&\sum_{s' \in \boldsymbol{S}} \hat{q}_{s'}(\underline{a}) u_2'(\omega^2_{ss'} + \hat{q}_{s'}(\underline{a}))\rho_{ss'} \\
&= \underline{a} \sum_{s' \in \boldsymbol{S}} \frac{u_2'(\omega^2_{ss'} + \hat{q}_{s'}(\underline{a}))}{u_1'(\omega^1_{s'} - \hat{q}_{s'}(\underline{a}))}\rho_{ss'} \\
&\geq \underline{a} \sum_{s' \in \boldsymbol{S}} \frac{u_2'(\omega^2_{ss'} + \underline{a}/u_1'(\omega^1_{s'}))}{u_1'(\omega_{s'} - \underline{a}/u_1'(\omega^1_{s'}))}\rho_{ss'} \\
&\geq \underline{a}\hat{z}(\underline{a}) \\
&\geq \underline{a},
\end{aligned}$$

where the first equality follows from the definition of $\hat{q}_{s'}(\,\cdot\,)$, the second inequality follows from Eq.(1), the third follows from the definition of $\hat{z}$, and the last follows from the fact that $\hat{z}(\underline{a}) \geq 1$. This implies that $\Phi(\underline{\boldsymbol{a}}) \geq \underline{\boldsymbol{a}}$. Since Φ is nondecreasing, it follows that $\Phi(\boldsymbol{a}) \geq \Phi(\underline{\boldsymbol{a}}) \geq \underline{\boldsymbol{a}}$ for all $\boldsymbol{a} \in \mathbf{R}^S_{++}$ with $\boldsymbol{a} \geq \underline{\boldsymbol{a}}$. This establishes the claim.

Let $\mathcal{A} := [\underline{a}, \overline{a}]^S$. $\mathcal{A}$ is obviously compact and convex and it is a subset of $\mathbf{R}^S_{++}$. Moreover, we have verified that Φ is continuous on $\mathcal{A}$ and $\Phi(\mathcal{A}) \subset \mathcal{A}$.

3. We apply the Brouwer fixed-point theorem to Φ on $\mathcal{A}$.[4] Since $\mathcal{A}$ is compact and convex, Φ is continuous, and $\Phi(\mathcal{A}) \subset \mathcal{A}$, it follows from the Brouwer fixed-point theorem that Φ has a fixed point $\boldsymbol{a}^* = (a^*_s)_{s \in \boldsymbol{S}}$ in $\mathcal{A}$, i.e., there exists some $\boldsymbol{a}^* \in \mathcal{A}$ such that $\Phi(\boldsymbol{a}^*) = \boldsymbol{a}^*$. Since $\boldsymbol{a}^* \in \mathbf{R}^S_{++}$, it follows that $\hat{q}_s(a^*_s) \in \text{int}.Q_s$ for all $s \in \boldsymbol{S}$.

4. We omit the proof of Step 4, since it is easy to verify that $\boldsymbol{q}^* := (\hat{q}_s(a^*_s))_{s \in \boldsymbol{S}}$ is a positive stationary equilibrium price vector. This establishes the proof of "existence" part of Proposition 1′.

(Uniqueness) Define the mapping $\Pi : \mathbf{R}^S_{++} \to \mathbf{R}^{S \times S}_{++}$ by $\Pi(\boldsymbol{q}) := [\pi_{ss'}(\boldsymbol{q})]_{s,s' \in \boldsymbol{S}}$, where

$$(\forall s, s' \in \boldsymbol{S})(\forall \boldsymbol{q} \in \mathbf{R}^S_{++}) \quad \pi_{ss'}(\boldsymbol{q}) := \frac{u_2'(\omega^2_{ss'} + q_{s'})\rho_{ss'}}{u_1'(\omega^1_s - q_s)}.$$

We have verified that at least one positive stationary equilibrium vector exists in the above argument. Notice that, for arbitrary positive stationary equilibrium price vector $\boldsymbol{q}^* \in \mathbf{R}^S_{++}$, we have the Frobenius root $\lambda_f(\Pi(\boldsymbol{q}^*)) = 1$, since $\Pi(\boldsymbol{q}^*)\boldsymbol{q}^* = \boldsymbol{q}^*$ (See the definition of equilibrium price vector). Also notice that $\boldsymbol{q}, \boldsymbol{q}' \in \mathbf{R}^S_{++}$ with $\boldsymbol{q} > \boldsymbol{q}'$, we have $\pi_{ss'}(\boldsymbol{q}) \leq \pi_{ss'}(\boldsymbol{q}')$ with at least one strict inequality. This follows immediately from concavity of u_i for $i = 1, 2$. The rest of the proof of the "uniqueness" part is the almost same as that of Magill and Quinzii (2003), so that see their work for more details. Q.E.D.

[4]One can apply the Tarski fixed-point theorem instead of the Brouwer fixed-point theorem, since $\mathcal{A}$ is a complete partial order set and Φ is nondecreasing. In this case, we no longer need to require continuity of Φ, i.e., continuity of ϕ and ψ. See, for example, Ok (2007) for the details of the Tarski fixed-point theorem.

4. Concluding Remarks

The proof of the existence of a stationary monetary equilibrium (SME) for a complicated overlapping generations (OLG) model with heterogeneous agents and several securities has been already provided. However, a simple and complete proof of the existence of SME for a simple, but rather canonical, model of monetary OLG economy presented in this paper has not provided yet. The main contribution of this paper is to provide a simple and complete proof of the existence of a SME for such a simple and canonical OLG model.

While previous studies applied the Frobenius theorem to show that all the prices are positive, this paper has shown it without the Frobenius theorem. This indicates that a simple OLG framework does not necessarily require the Frobenius theorem to ensure that all the prices are positive.

Also remark that we can no longer show that all the prices are positive by the Frobenius theorem when the amount of the second-period endowment is zero in some state, i.e., when $\omega^2_{ss'} = 0$ for some $s, s' \in \boldsymbol{S}$, since the matrix Π^0 is no longer well-defined under Assumption 3.1. In such cases, therefore, we can no longer verify that prices of money are positive by a similar procedure to the one adopted by previous studies. On the other hand, this paper provided the existence result that includes such cases. This indicates that this paper has shown the existence of a SME for a broader range of models than that of previous studies such as Magill and Quinzii (2003).

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