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Power of the KPSS test against shift in variance: a further investigation.

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Abstract

This paper shows some failures of the KPSS test when the source of the nonstationarity is explained by an unconditional volatility shift. We provide the asymptotic moments of the statistic under general case of shifts in the unconditional variance. We find that these moments remain unchanged even under high abrupt changes. Finally a complementary test is proposed.

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1 Introduction

The KPSS test is often used for testing the null hypothesis of stationarity against the alternative of unit root. This test is among the most used by practioners since they are implemented in many software. But as noted by some authors, the non-rejection of the null hypothesis does not imply often the stationarity of the data (Ahamada, 2004). When the source of the nonstationarity of the data is concerned with a shift in the unconditional volatility instead of unit root, then the KPSS test fails to detect this form of instability, the null is not rejected while the process is not really covariance-stationary. These failures about the KPSS test were demonstrated from monte-carlo experiments only. But no theoretical considerations was proposed to explain these findings. This paper provides a theoretical explanation. We compute the exact asymptotic moments of the KPSS test under shift in the unconditional variance. We show that these asymptotic moments remain unchanged even under high abrupt changes. These findings give a theoretical explanation of earlier results based only on Monte-Carlo simulations. This paper is organised as follows: section two presents the asymptotic moments of the KPSS test under changes in the variance. Since the KPSS test fails to detect variance shift, we propose also a complementary test in this same section. Some simulation experiments are presented in section three. The last section concludes the paper.

2 Asymptotic results and complementary test.

The KPSS test is based on the following model:

$$y_t = r_t + \varepsilon_t,\ t = 1, ..., T \tag{1}$$

where $r_t = r_{t-1} + u_t$ is a random walk and ε_t is a stationary process with the following assumptions :

Assumptions : *The* ε_t's *and* u_t's *are mutually independent normal and i.i.d with* $E(\varepsilon_t) = 0$, $var(\varepsilon_t) = \sigma_\varepsilon^2$, $E(u_t) = 0$ *and* $var(u_t) = \sigma_u^2 \geqslant 0$.

The null hypothesis is $H_0 : E(u_t^2) = \sigma_u^2 = 0$ which means that the component r_t is a constant instead of a unit root process. Under the null hypothesis, y_t is stationary around level r_0. The alternative hypothesis is given by, $H_1 : \sigma_u^2 > 0$. Let us consider the statistic:

$$\widehat{\eta}_{\mu T} = \frac{T^{-2} \sum_{t=1}^{T} S_t^2}{\widehat{\sigma}^2} \tag{2}$$

where $S_t = \sum_{j=1}^{t} \widehat{e}_j$, $\widehat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} \widehat{e}_t^2$, and $\widehat{e}_t$'s are the residuals from regression $y_t = r_0 + \varepsilon_t$. Let $m_T = E(\widehat{\eta}_{\mu T})$ and $\sigma_T^2 = var(\widehat{\eta}_{\mu T})$. Under the null hypothesis of stationarity around the level r_0 the limiting distribution of $\widehat{\eta}_{\mu T}$ is given by $\int V(r)^2 dr$ where $V(r) = W(r) - rW(1)$ and $W(r), r \in [0, 1]$, is a Brownian motion process. For small values of T one can use numerical methods to compute m_T and σ_T^2 (see section A of appendix). What are the values of m_T and σ_T^2 when the variance of ε_t is time varying[1] in(1)? To answer to this question we consider the following process:

$$y_t = r_t + h_t \varepsilon_t,\ t = 1, ..., T \tag{3}$$

where the sequence (h_t) is a bounded deterministic sequence allowing heteroskedasticity in residuals: $var(h_t \varepsilon_t) = \sigma_\varepsilon^2 h_t^2$. Without loss of generality one can always set $var(\varepsilon_t) = \sigma_\varepsilon^2 = 1$ in (3).

Lemma. Let $\widehat{\sigma}_T^2 = \sum_{t=1}^{T} e_t^2 / T$ and e_t the residuals from the regression: $y_t = r_0 + h_t \varepsilon_t, \varepsilon_t \sim i.i.d.N(0, 1)$ where (h_t) is a bounded deterministic sequence such that

$$\sum_{t=1}^{T} h_t^2 / T \rightarrow \overline{h}_2^2 \text{ as } T \rightarrow \infty \tag{4}$$

then

$$\frac{1}{T} \sum_{t=1}^{T} e_t^2 \quad \xrightarrow{a.s.} \quad \overline{h}_2^2. \tag{5}$$

Proof: See section B of appendix.

Theorem. *Assume in model (3) that* (h_t) is a bounded deterministic sequence with the

[1]For example, when the variance is affected by an abrupt change.

condition (4).

Then under the null hypothesis that there is no unit root component in model (3) the following results are valid:

$$\lim_{T\to\infty} m_T = \frac{1}{6} \tag{j}$$

and

$$\lim_{T\to\infty} \sigma_T^2 = \frac{1}{45}. \tag{jj}$$

Proof: See section C of appendix.

The theorem shows that under general conditions in h_t, the asymptotic moments of the statistic of the KPSS test remain the same as in the case where h_t is constant.

These results allow to understand and to complet earlier results based only on Monte-Carlo simulations (Ahamada, 2004). The Monte-Carlo simulations have allowed to conclude that when the source of the nonstationarity of the data is concerned with a shift in the unconditional volatility instead of unit root, then the KPSS test fails to detect this form of instability. The null is not rejected while the process is not really stationary. The stability of the moments (even under strong break as it is indicated by the theorem) contributes to understand this lack of power. Further simulations are proposed in section three.

2.1 A complementary test

Since the KPSS test fails to detect bounded shifts in variance, we propose in this subsection a complementary test. We are concerned with a test of the null hypothesis of variance constancy in (3): $H_0^{(2)} : h_t = constant$. Let us consider the statistic τ defined as follows:

$$\tau = \max_{k=1,\dots,T} \sqrt{\frac{T}{2}} \left| D_k \right|,$$

where $D_k = \frac{C_k}{C_T} - \frac{k}{T}$, $C_k = \sum_{j=1}^{k} \widehat{e}_j^2$, $k = 1, ..., T$ and $\widehat{e}_j$ the same residuals used for the computation of the $\widehat{\eta}_{\mu T}$ in (2).

Proposition. Under the null hypothesis of variance constancy,.i.e. $H_0^{(2)}$, and *assuming that* ε_t *is* $i.i.N$ $(0,\ \sigma_\varepsilon^2)$, then the limiting distribution of τ is given by $sup\,(W_t^0)$ where W_t^0 is a standard Brownian Bridge.

Proof. Under the null hypothesis $H_0^{(2)}$ and *assuming that* ε_t *is* $i.i.N$ $(0,\ \sigma_\varepsilon^2)$ then from regression (1): $var(\ \widehat{e}_t\ ) = \sigma_\varepsilon^2 + T^{-1}\sigma_\varepsilon^2$ and $cov(\widehat{e}_t\ ,\widehat{e}_{t'}\ ) = \min(t,t')T^{-2}\sigma_\varepsilon^2$. Hence $\widehat{e}_t$ are asymptotically $i.i.N$ $(0,\ \sigma_\varepsilon^2)$. So the condition of the theorem of Inclan and Tiao(1994) is obviously satisfied. The limiting distribution of $\sqrt{\frac{T}{2}}\,|D_k|$ is given by the one of W_t^0 where W_t^0 is a standard Brownian Bridge. So the $\max\limits_{k=1,\ldots,T}\sqrt{\frac{T}{2}}\,|D_k|$ is asymptotically distributed as $sup\,(W_t^0)$ and the desired conclusion holds. From Inclan and Tiao (1994), $C_{0.05} = 1.36$ with $\Pr[\sup\,(W_t^0) > C_{0.05}] = 0.05$.

We suggest to use the complementary test as follows: First, apply the KPSS test. If the null hypothesis is rejected, then conclude that the data contain a unit root, i.e. there is nonstationarity. If the null is not rejected, then there is no unit root but there are a possible shifts in the variance. Then apply the statistic τ. If the statistic τ does not reject the null hypothesis, then there is covariance-stationarity. If the null is rejected by τ, then conclude that there is no unit root but data have variance shift and the process is not covariance-stationary.

3 Monte-Carlo experiments

We consider the following data-generating process (DGP)

$$y_t = 0.01 + h_t\varepsilon_t, \quad t = 1,...,200 \ \text{ and } \varepsilon_t \text{ is } i.i.d.N(0,1)$$

where $h_t = \sigma_1 > 0$ if $t = 1,...,100$ and $h_t = \sigma_2 > 0$ if $t = 101,...,200$. The ratio $\vartheta = \frac{\sigma_2}{\sigma_1}$ gives the size of the shift in the unconditional variance. When $\vartheta = 1$ the null hypothesis of covariance stationarity holds (no unit root and no variance shift). The ε_t's are generated from the standard $N(0,1)$. For each value of ϑ, we generate 1000 values of $\widehat{\eta}_{\mu T}$ and τ. We compute the proportion of the rejection of the null hypothesis for both tests. We use asymptotic critical values at the 0.05 level ($C_{0.05}(\widehat{\eta}_{\mu T}) = 0.463$ and $C_{0.05}(\tau) = 1.360$). Table

1 reports results about the proportion of the rejection of the null hypothesis. For the KPSS test, the values remain always near the nominal size $\alpha = 0.05$. This also remains true when there is increase in the size of the shift in the unconditional variances (i.e. increase in $\vartheta = \frac{\sigma_2}{\sigma_1}$). These results confirm the theoretical finding of the theorem 1. The statistic $\widehat{\eta}_{\mu T}$ behaves as in the case of the stationary model (i.e., $\vartheta = 1$) even when high abrupt change is hiden in the data. For the Statistic τ, we can see that the power increases with the size of the shift ϑ. Hence the choice of τ as complementary test seems to be credible.

Table 1. Power under shift in variance ($\vartheta = \frac{\sigma_2}{\sigma_1}$)

	$\vartheta = \mathbf{1}$	$\vartheta = \mathbf{2}$	$\vartheta = \mathbf{5}$	$\vartheta = \mathbf{10}$	$\vartheta = \mathbf{15}$	$\vartheta = \mathbf{20}$
Statistic τ	0.046	0.96	0.97	0.99	1.00	1.00
KPSS	0.048	0.049	0.052	0.053	0.052	0.053

4 Conclusion.

In this paper we focused on the KPSS test when changes occur in the variance of errors. We have shown that the asymptotic moments remain unchanged even under strong abrupt changes. These theoretical results allow to complete earlier findings based only on Monte-Carlo experiments. Since many financial and macro-economic time series are characterized by changes in the unconditional variance, some precaution must be taken when using the KPSS test to check the stationarity of second order moment. A complementary test is to investigate possible breaks in the unconditioanal variance.

APPENDIX

A. Computation of $m_T = E(\widehat{\eta}_{\mu T})$ and $\sigma_T^2 = var(\widehat{\eta}_{\mu T})$.

Without loss of generality we assume that the intercept is zero in regression $y_t = r_0 + \varepsilon_t$. The residuals are $e_t = \varepsilon_t - \overline{\varepsilon}, \overline{\varepsilon} = \sum_{t=1}^{T} \varepsilon_t / T$. It can be shown that $\widehat{\eta}_{\mu T}$ can be written as the

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ratio of quadratic form in $\underline{\varepsilon} = (\varepsilon_1, ..., \varepsilon_T)'$, $i.e.$

$$\widehat{\eta}_{\mu T} = T^{-1}\frac{\underline{\varepsilon}'C'AC\underline{\varepsilon}}{\underline{\varepsilon}'C\underline{\varepsilon}}$$

where $C = I_T - \underline{1}\underline{1}'T^{-1}$, I_T is the identity matrix of dimension T and $\underline{1}$ is a T-dimensional vector of ones, and $\underline{\varepsilon} \sim N(0, \Gamma)$, $\Gamma = \text{diag}(h_1^2, ..., h_T^2)$. Let Q be an orthogonal matrix ($i.e.Q'Q = I_T$) such that

$$Q\Gamma^{1/2}C\Gamma^{1/2}Q = D = diag(d_1, ..., d_T),$$

where $\Gamma^{1/2} = \text{diag}(h_1, ..., h_T)$ and let $\Lambda = Q'\Gamma^{1/2}C\Gamma^{1/2}Q = (\lambda_{i,j})$ then the exact moments of $\widehat{\eta}_{\mu T}$ (see Jones (1987)) are given by

$$\begin{aligned} E(\widehat{\eta}_{\mu T}) &= m_T = T^{-1}\int_0^{+\infty}\sum_{i=1}^{T}\frac{\lambda_{i,i}}{1+2d_it}\Phi(t,d)dt, \qquad (6)\\ E(\widehat{\eta}_{\mu T}^2) &= T^{-2}\int_0^{+\infty}\sum_{i=1}^{T}\sum_{j=1}^{T}\frac{\lambda_{i,i}\lambda_{j,j}+2\lambda_{i,j}}{(1+2d_it)(1+2d_jt)}\Phi(t,d)dt, \end{aligned}$$

where $\Phi(t,d) = \prod_{i=1}^{T}(1+2d_it)^{-1/2}$. For small T, one can use the numerical methods proposed by Paolella (2003) to compute the moments. But for large T such methods are time consuming and the asymptotic values of $E(\widehat{\eta}_{\mu T})$ and $E(\widehat{\eta}_{\mu T}^2)$ will be useful.
To prove (j) and (jj) of the theorem we need to prove the Lemma.

B. Proof of the Lemma.

Let $\widehat{\sigma}_T^2 = \sum_{t=1}^{T} e_t^2/T$ and e_t are the residuals from regression: $y_t = r_0 + h_t\varepsilon_t$. Assume that $r_0 = 0$ without loss of generality. We have $e_t = h_t\varepsilon_t - \overline{h\varepsilon}, \overline{h\varepsilon} = \sum_{t=1}^{T} h_t\varepsilon_t/T$ hence

$$\frac{1}{T}\sum_{t=1}^{T} e_t^2 = \frac{1}{T}\sum_{t=1}^{T}\ (h_t\varepsilon_t)^2 - \overline{(h\varepsilon)}^2 \qquad (7)$$

$M_T = \sum_{t=1}^{T} h_t\varepsilon_t$ is a square integrable martingale adapted to the $\sigma-$field $\mathcal{F}_T = \sigma(\varepsilon_1, ..., \varepsilon_T)$ with the increasing process

$$\begin{aligned} \langle M_T \rangle &= \sum_{t=1}^{T} E((h_t\varepsilon_t)^2 \mid \mathcal{F}_{t-1}) \\ &= \sum_{t=1}^{T} h_t^2 \ \to \infty \end{aligned}$$

The application of theorem 1.3.15 of Duflo (2003) leads to $M_T/ < M_T > \to 0$ almost surely, this with (4) imply that

$$\overline{h\varepsilon} = \frac{1}{T}\sum_{t=1}^{T} h_t\varepsilon_t \quad \xrightarrow{a.s.} \quad 0 \tag{8}$$

Likewise $M_T =_{t=1}^{T} h_t^2(\varepsilon_t^2 - 1)$ is a square integrable martingale adapted to $\digamma$, with increasing process $\langle M_T \rangle = 2_{t=1}^{T} h_t^4$, hence theorem 1.3.15. in Duflo (2003) implies that

$$\frac{1}{\langle M_T \rangle}{}_{t=1}^{T} h_t^2(\varepsilon_t^2 - 1) \longrightarrow 0 \text{ almost surely on } \{\langle M_\infty \rangle = \infty\} \tag{9}$$

where $\langle M_\infty \rangle = \lim_{T\to\infty} \langle M_T \rangle$. Since (Cauchy Schwarz inequality)

$$\left(\sum_{t=1}^{T} h_t^2\right)^2 \leq T_{t=1}^{T} h_t^4, \tag{10}$$

The assumption (4) implies that there exist an universal constants $0 < K_1 < K_2 < \infty$ such that

$$K_1 < \frac{1}{T}\sum_{t=1}^{T} h_t^2 < K_2,$$

this together with (10) implies that $\langle M_T \rangle \geq 2TK_1^2$ which implies that $\{\langle M_\infty \rangle = \infty\} = \Omega$ and hence

$$\frac{1}{\langle M_T \rangle}{}_{t=1}^{T} h_t^2(\varepsilon_t^2 - 1) \xrightarrow{a.s.} 0, \tag{11}$$

Since (h_t) is a bounded deterministic sequence, then there exists an universal $K > 0$ such that $h_t^4 \leq K$ for all $t \geq 1$, hence $\langle M_T \rangle \leq 2TK$ for all T, therefore

$$\begin{aligned} \left|\frac{1}{T}{}_{t=1}^{T} h_t^2(\varepsilon_t^2 - 1)\right| &= \frac{\langle M_T \rangle}{T}\left|\frac{1}{\langle M_T \rangle}{}_{t=1}^{T} h_t^2(\varepsilon_t^2 - 1)\right| \\ &\leq 2K\left|\frac{1}{\langle M_T \rangle}{}_{t=1}^{T} h_t^2(\varepsilon_t^2 - 1)\right|, \end{aligned}$$

using (11), it follows that

$$\frac{1}{T}{}_{t=1}^{T} h_t^2(\varepsilon_t^2 - 1) \xrightarrow{a.s.} 0. \tag{12}$$

Combining (4) and (7), (8) and (12) we obtain (5).

C. Proof of the theorem.

From lemma we deduce that $\widehat{\eta}_{\mu T}$ has the same limiting distribution as

$$\widehat{\psi}_{\mu T}=\frac{1}{T^2\overline{h}_2^2}\sum_{t=1}^{T}S_t^2,S_t=\sum_{j=1}^{t}e_j, \tag{13}$$

and hence

$$\lim_{T\to\infty}m_T=\lim_{T\to\infty}E(\widehat{\psi}_{\mu T})\text{ and }\lim_{T\to\infty}\sigma_T^2=\lim_{T\to\infty}\operatorname{var}(\widehat{\psi}_{\mu T}). \tag{14}$$

Let D_t the $(2,T)$ matrix given by

$$D_t=\begin{pmatrix}\frac{t}{T} & \cdots & \cdots & \cdots & \cdots & \frac{t}{T}\\ 1 & \cdots & 1 & 0 & \cdots & 0\end{pmatrix}$$

the ones are repeated t times. Since $S_t=e_1D_t\underline{u}$,where $e_1=(-1,1)',\underline{u}=(u_1,...,u_T)'$ $u_t=h_t\varepsilon_t$, S_t^2 can be written as a quadratic form in $\underline{u}$, $S_t^2=\underline{u}'D_t'e_1e_1'D_t\underline{u}$. Consequently (see Magnus (1986))

$$\begin{aligned}E(S_t^2) &= \operatorname{trace}(D_t'e_1e_1'D_t\Gamma),\Gamma=diag(h_1^2,...,h_T^2)\\ &= \operatorname{trace}(\Gamma^{1/2}D_t'e_1e_1'D_t\Gamma^{1/2})\\ &= \left\|\Gamma^{1/2}D_t'e_1\right\|^2 ,\end{aligned}$$

where $\|x\|^2$ is the Euclidian norm of the vector x.

Now

$$\begin{aligned}\left\|\Gamma^{1/2}D_t'e_1\right\|^2 &= \sum_{j=1}^{t}h_j^2\left(1-\frac{t}{T}\right)^2+\sum_{j=t+1}^{T}h_j^2\left(\frac{t}{T}\right)^2\\ &= \left(\sum_{j=1}^{T}h_j^2\right)\left(\frac{t}{T}\right)^2+\sum_{j=1}^{t}h_j^2\left(1-\frac{2t}{T}\right)\end{aligned} \tag{15}$$

Hence

$$\begin{aligned}E(\widehat{\psi}_{\mu T}) &= \frac{1}{T^2\overline{h}_2^2}\sum_{t=1}^{T}E(S_t^2)\\ &= \frac{1}{T^2\overline{h}_2^2}\left\{\sum_{t=1}^{T}\left(\sum_{j=1}^{T}h_j^2\right)\left(\frac{t}{T}\right)^2+\sum_{j=1}^{t}h_j^2\left(1-\frac{2t}{T}\right)\right\}\\ &= \frac{1}{\overline{h}_2^2}\left\{\left(\frac{1}{T}\sum_{j=1}^{T}h_j^2\right)\frac{1}{T^3}\sum_{t=1}^{T}t^2+\frac{1}{T^2}\sum_{t=1}^{T}\sum_{j=1}^{t}h_j^2-\frac{2}{T^3}\sum_{t=1}^{T}t\sum_{j=1}^{t}h_j^2\right\}\end{aligned}$$

The convergence (4) implies (see lemma 2 of Boutahar and Deniau (1996)) that

$$\begin{aligned} \frac{1}{T^2}\sum_{t=1}^{T}\sum_{j=1}^{t} h_j^2 &\to \frac{\overline{h}_2^2}{2} \\ \frac{1}{T^3}\sum_{t=1}^{T} t\sum_{j=1}^{t} h_j^2 &\to \frac{\overline{h}_2^2}{3} \end{aligned}$$

Therefore

$$\lim_{T\to\infty} E(\widehat{\psi}_{\mu T}) = \frac{1}{3} + \frac{1}{2} - \frac{2}{3} = \frac{1}{6} \tag{16}$$

and (j) follows from (14) and (16).

$$\text{var}\left(\frac{1}{T^2}\sum_{t=1}^{T} S_t^2\right) = \frac{1}{T^4}\left\{\sum_{t=1}^{T}\text{var}(S_t^2) + 2\sum_{s<t}\text{cov}\left(S_s^2, S_t^2\right)\right\}. \tag{17}$$

Let $\Lambda_t = D_t' e_1 e_1' D_t'$, and $\Gamma^{1/2} = diag(h_1, ..., h_T)$, we have

$$\begin{aligned} \text{var}(S_t^2) &= 2\text{trace}\left(\Lambda_t \Gamma \Lambda_t \Gamma\right) \\ &= 2\text{trace}\left(D_t' e_1 e_1' D_t' \Gamma D_t' e_1 e_1' D_t' \Gamma\right) \\ &= 2\text{trace}\left(\left(\Gamma^{1/2} D_t' e_1 e_1' D_t' \Gamma^{1/2}\right)\left(\Gamma^{1/2} D_t' e_1 e_1' D_t' \Gamma^{1/2}\right)\right) \\ &= 2\left\|\Gamma^{1/2} D_t' e_1\right\|^4. \end{aligned}$$

By using (15) we get

$$\text{var}(S_t^2) = 2\left[\left(\sum_{j=1}^{T} h_j^2\right)\left(\frac{t}{T}\right)^2 + \sum_{j=1}^{t} h_j^2\left(1 - \frac{2t}{T}\right)\right]^2$$

By using (4), a straightforward computation leads to

$$\frac{1}{T^4}\sum_{t=1}^{T}\text{var}(S_t^2) \sim \frac{(\overline{h}_2^2)^2}{15T} = \frac{\overline{h}_2^4}{15T}, \tag{18}$$

where $a_T \sim b_T$ means that $a_T/b_T \to 1$ as $T \to \infty$. For $s < t$

$$\begin{aligned} \text{cov}\left(S_s^2, S_t^2\right) &= 2\text{trace}\left(\Lambda_s \Gamma \Lambda_t \Gamma\right) \\ &= 2\text{trace}\left(D_s' e_1 e_1' D_s' \Gamma D_t' e_1 e_1' D_t' \Gamma\right) \\ &= 2\text{trace}\left(\left(\Gamma^{1/2} D_s' e_1 e_1' D_s' \Gamma^{1/2}\right)\left(\Gamma^{1/2} D_t' e_1 e_1' D_t' \Gamma^{1/2}\right)\right) \\ &= 2\text{trace}\left((xx')(yy')\right) \\ &= 2(x'y)^2, \end{aligned}$$

where $x = \Gamma^{1/2} D_t' e_1, y = \Gamma^{1/2} D_s' e_1$.

Since

$$x' = (h_1(1-t/T), ..., h_s(1-t/T), ..., h_t(1-t/T), -h_{t+1}t/T, ..., -h_T t/T)$$

and

$$y' = (h_1(1-s/T), ..., h_s(1-s/T), -h_{s+1}s/T, ..., -h_T s/T) \, .$$

Therefore

$$\operatorname{cov}\left(S_s^2, S_t^2\right) = 2\left\{\sum_{j=1}^{s} h_j^2 \left(1-\frac{s}{T}\right)\left(1-\frac{t}{T}\right) - \sum_{j=s+1}^{t} \frac{s}{T}\left(1-\frac{t}{T}\right) + \sum_{j=t+1}^{T} h_j^2 \frac{ts}{T^2}\right\}^2 .$$

By using (4), a straightforward computation leads to

$$\begin{aligned} \frac{1}{T^4}\sum_{s<t} \operatorname{cov}\left(S_s^2, S_t^2\right) &\sim \frac{2\overline{h}_2^4}{T^4}\sum_{s=1}^{T}\sum_{t=s+1}^{T}\left\{\frac{st}{T} + s\left(1-\frac{(s+t)}{T}\right) - (t-s)\frac{s}{T}\right\} \qquad (19) \\ &\sim \frac{\overline{h}_2^4}{90}, \end{aligned}$$

since

$$\begin{aligned} \sum_{s=1}^{T}\sum_{t=s+1}^{T}(ts)^2 &\sim \frac{T^6}{18}, \\ \sum_{s=1}^{T}\sum_{t=s+1}^{T} ts^2 &\sim \frac{T^5}{15}, \\ \sum_{s=1}^{T}\sum_{t=s+1}^{T} s^2 &\sim \frac{T^4}{12}. \end{aligned}$$

From (13), (17), (18) and (19) we deduce that

$$\begin{aligned} var(\widehat{\psi}_{\mu T}) &= \frac{1}{\overline{h}_2^4} \operatorname{var}\left(\frac{1}{T^2}\sum_{t=1}^{T} S_t^2\right) \\ &\sim \frac{1}{45}, \end{aligned}$$

from this and (14), (jj) holds.

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