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Lower and upper tail concern and the rank dependent social evaluation functions

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Abstract

Some properties that strengthen the Pigou-Dalton transfer principle have been introduced in the literature, such as the diminishing transfer principle, and its positional version, the principle of positional transfer sensitivity. These principles state that social evaluation function should be more sensitive to transfers that take place lower down in the distribution and reflect a lower tail concern. Another principle which operates in the opposite direction is also introduced, first degree upside positional transfer sensitivity which imposes more sensitivity to transfers further up the scale; and therefore reflects an upper tail concern. In this paper a new principle is proposed, up-down positional transfer sensitivity, which combines the lower tail concern when transfers occur at the low tail of the distribution, and the upper tail concern when they happen at the up tail. Some rationales behind the new property and its implications for the class of rank dependent social evaluation functions are analyzed. It is shown that the Gini function satisfies this property, and a class of social evaluation functions consistent with it is proposed.

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1. Introduction

Social welfare increases under a transfer from a richer to a poorer person; this idea is captured by the Pigou-Dalton transfer principle. However, by how much different transfers increase social welfare is very much less clear. In this sense, Kolm (1976a,b) argues that the Pigou-Dalton transfer principle can be strengthened by imposing that social welfare should be more sensitive to transfers that take place lower down in the distribution, and suggests the diminishing transfer principle.[1] This states that, for a fixed income gap, a transfer of the same amount from a richer to a poorer person should be considered to increase social welfare more the lower it occurs in the distribution. In the limiting case, the only transfers which increase social welfare are transfers from any individual to the poorest one. Mehran (1976) and Kakwani (1980) introduce a positional version of this principle, the principle of positional transfer sensitivity, henceforth PTS, which demands that, instead of the income gap, the proportion of the population is fixed.[2]

Aaberge (2009) introduces the principle of first degree upside positional transfer sensitivity, henceforth UPTS, which demands that a transfer of the same amount from a richer to a poorer person should be considered to increase social welfare more, the richer the donor is, provided that the population between them is fixed. Thus, it can be considered as the "dual" of PTS since both PTS and UPTS perform in opposite directions. UPTS emphasises the social welfare effect of transfers occurring higher up in the distribution. In the limiting case, the only transfers which increase social welfare would be transfers from the richest individual to anyone else in the society.

Actually, in each transfer there are always two agents, the donor and the recipient. What we propose is that when the transfer takes place below some threshold, our focus would be on the recipient, and hence the poorer the recipient is, the better. Whereas above this threshold we would look at the donor, and consequently the richer the donor is, the better. It is worth noting that of the existing principles, neither PTS nor UPTS, manage to capture this idea. Therefore a new principle, imposing different conditions for transfers taking place above or below some threshold, may have some room in this literature and something new to offer over the axioms already proposed.

We illustrate the intuitions behind the new principle with the following examples. First, let $x=(a,b,c,d,e,f,g,h)$ be an income distribution in an eight individual society such that $a<b<c<d<e<f<g<h$, and consider two transfers leading to $y^1=(a,b,c,d,e,f,g+\Delta,h-\Delta)$ such that $g<g+\Delta\leq h-\Delta<h$, and $y^2=(a,b,c,d,e+\Delta,f-\Delta,g,h)$ such that $e<e+\Delta\leq f-\Delta<f$. Since all the individuals involved in the transfers are at the upper tail of the distribution, the new principle demands that social welfare increases more under the first transfer, that affecting the richest. It is worth noting that UPTS ranks these two distributions in this way, but PTS does just the opposite. Whereas if we consider two transfers at the lower tail leading to $y^3=(a,b,c+\Delta,d-\Delta,e,f,g,h)$ such that $c<c+\Delta\leq d-\Delta<d$, and $y^4=(a+\Delta,b-\Delta,c,d,e,f,g,h)$ such that $a<a+\Delta\leq b-\Delta<b$, respectively, the new principle states that social welfare should increase more after the second transfer, that affecting the poorest. In this case, PTS ranks these two distributions in this way. By contrast UPTS ranks them in the opposite direction. Neither PTS nor UPTS ranks these four distributions, y^1, y^2, y^3 and y^4, in the way proposed by the new principle.

[1] Davies and Hoy (1995) call this axiom aversion to downside inequality, and Shorrocks and Foster (1987) introduces a similar axiom called transfer sensitivity.
[2] Aaberge (2009) refers to it as the principle of first degree downside positional transfer sensitivity.

Consider now the following distributions, $z^1 = (a+\Delta, b-\Delta, c, d, e, f, g+\Delta, h-\Delta)$ where $a < a+\Delta \le b-\Delta < b < g < g+\Delta \le h-\Delta < h$, and $z^2 = (a, b, c+\Delta, d-\Delta, e+\Delta, f-\Delta, g, h)$ where $c < c+\Delta \le d-\Delta < d < e < e+\Delta \le f-\Delta < f$. Notice that distribution z^1 can be derived from x by two progressive transfers occurring at both ends of the distribution, while two progressive transfers taking place in the centre of the distribution are needed to convert x into z^2, the amounts transferred being the same in both cases. The comparisons between x and z^1 and between x and z^2 are uncontroversial for every measure fulfilling the Pigou Dalton transfer principle. This is not the case for the comparison between z^1 and z^2. PTS and UDPTS do not establish how to order these two distributions. We consider that there are reasons to state that the increase in social welfare from x to z^1 is greater than the increase from x to z^2. First, if we look at incomes at the low tail of the distribution, we observe that in z^1 the recipient is poorer than in z^2. In z^1, the poorest individual is the one who receives the transfer and is now better off. Secondly, if we focus on incomes at the up tail, we observe that in z^1 the donor is richer than in z^2, and the one who transfers is the richest individual. Finally, the most extreme incomes are closer to the median in distribution z^1 than in z^2. The purpose of this paper is to introduce a new principle that allows us to compare this type of distribution.

In this paper we focus on rank dependent social evaluation functions that depend not only on income shares but also on ranks. We introduce a principle referred to as up-down positional transfer sensitivity, henceforth UDPTS. For this, the population is split into two groups according to some threshold, and for incomes below it UDPTS behaves like PTS; and for incomes above like UPTS. In particular in this paper, we have chosen the median as the threshold, but similar results can be derived using the same machinery when thresholds different from the median are selected.

Section 3 analyzes some implications of UDPTS. The Gini social evaluation function attaches an equal weight to a transfer irrespective of where it takes place in the income distribution, as long as the transfer occurs between individuals with the same difference in ranks. Thus, the Gini social evaluation function satisfies both PTS and UPTS, and consequently UDPTS. In Section 4 we propose a family of measures which contains the Gini function and fulfils this new principle. Section 2 introduces the notation and some basic concepts. Section 5 concludes.

2. Notation and preliminaries

We assume throughout that incomes are drawn from an interval D which is a compact subset of $\mathbb{R}$. An income distribution for a population consisting of *n* identical individuals $(n \ge 2)$ is a list $x := (x_1, x_2, ..., x_n)$ such that $x_1 \le x_2 \le ... \le x_n$, where $x_i \in D$ is the income of individual *i*. We denote $Y(D)$ the set of all distribution for a population of size *n*. Let $\mu(x)$ be the *mean* of $x \in Y(D)$, that is $\mu(x) := \sum_{i=1}^{n} x_i / n$.

A *social evaluation function* $W : Y(D) \to \mathbb{R}$ associates to every distribution a real number $W(x)$ that represents the social welfare attained in income distribution $x \in Y(D)$. When $W(x) \ge W(y)$, then we will say that distribution *x* is at least as good as distribution *y*. Although the focus of our concern is to make welfare and inequality comparisons of

distributions whose sizes may differ, assuming the principle of population there is no loss of generality restricting attention to distributions with the same size, *n*. [3]
Let $F(\cdot, x)$ be the cumulative distribution of $x \in Y(D)$ define by $F(z,x) = q(z,x)/n$, for all $z \in (-\infty, \infty)$, where $q(z,x) = \#\{i \in \{1,2,...,n\} / x_i < z\}$. The left inverse cumulative distribution $F^{-1}(\cdot, x)$ represents the quantile function of *x*. Then $F^{-1}(t,x)$ shows the income received by an individual with rank *t*, being $F^{-1}(0,x) = x_1$. In this paper, for the sake of simplicity and whenever it is not confusing, we will denote $F^{-1}(t,x)$ as $F^{-1}(t)$.

For a social evaluation function W, any income distribution $x \in Y(D)$, and for $\delta > 0$, we shall denote by $\Delta W(t,\delta,\rho)$ the change in *W* resulting from a transfer δ from the person with income $F^{-1}(t)$ to the person with income $F^{-1}(t-\rho)$ that leaves their rank in the distribution unchanged, that is

$$\Delta W(t,\delta,\rho) := W\left(..., F^{-1}(t-\rho)+\delta, ..., F^{-1}(t)-\delta, ...\right) - W\left(..., F^{-1}(t-\rho), ..., F^{-1}(t), ...\right)$$

Such a transfer is called a *progressive transfer*.[4]

The Pigou- Dalton transfer principle, PD, states that any progressive transfer increases social welfare.[5] More formally,

Definition 1. Pigou-Dalton transfer principle, PD. A social evaluation function *W* is said to satisfy the Pigou-Dalton transfer principle, if for any individual with income distribution $x \in Y(D)$, $\Delta W(t,\delta,\rho) > 0$.

Positional transfer sensitivity, introduced and analyzed by Mehran (1976) and Kakwani (1980), demands that a rank preserving transfer from a richer to a poorer person increases social welfare by a higher amount the lower it occurs in the distribution, provided that the number of individuals between the donor and the recipient is given.[6]

Definition 2. Positional transfers sensitivity, PTS. A social evaluation function *W* is said to satisfy positional transfers sensitivity, if for any pair of individuals with incomes $F^{-1}(i)$ and $F^{-1}(j)$, $\Delta W(i,\delta,\rho) \geq \Delta W(j,\delta,\rho)$ when $i < j$.

Aaberge (2009) introduces the principle of first degree upside positional transfer sensitivity, which in contrast to PTS, imposes that social evaluation be more sensitive to transfers that take place further up in the distribution. [7]

Definition 3. First degree upside positional transfer sensitivity, UPTS. A social evaluation function W is said to satisfy first degree upside positional transfer sensitivity, if for any pair of individuals with incomes $F^{-1}(i)$ and $F^{-1}(j)$, $\Delta W(i,\delta,\rho) \leq \Delta W(j,\delta,\rho)$ when $i < j$.

The principles PTS and UPTS, operate in opposite directions. The choice between PTS and UPTS makes it clear whether the focus is on transfers that take place in the lower or

[3] The principle of population states that a replication does not affect social welfare.
[4] For the definition of a regressive transfer, both the donor and the recipient change their role, or δ is considered negative.
[5] See Fields and Fei (1978).
[6] If we come back to the example in the Introduction section, notice that positional transfer sensitivity demands that $W(y^4) \geq W(y^3) \geq W(y^2) \geq W(y^1)$.
[7] If we come back to the example in the Introduction section, notice that first degree upside positional transfer sensitivity demands that $W(y^4) \leq W(y^3) \leq W(y^2) \leq W(y^1)$.

upper part of the income distribution. Notice that both PTS and UPTS impose their requirements on the whole distribution.

We will focus on rank dependent social evaluation functions[8] defined by

$$W_\psi(x) := \sum_{i=1}^{n}\left[\psi\left(\frac{n-i+1}{n}\right)-\psi\left(\frac{n-i}{n}\right)\right]x_i\,, \tag{1}$$

where $\psi \in \boldsymbol{\psi} := \{\psi : [0,1] \to [0,1] / \psi$ is non decreasing, $\psi(0)=0$ and $\psi(1)=1\}$

Equation (1) can be equivalently rewritten as[9]

$$W_\psi(x) = \sum_{i=1}^{n}\psi'\left(\frac{n-i}{n}\right)x_i \tag{2}$$

where $\psi'\left(\frac{n-i}{n}\right) := \psi\left(\frac{n-i+1}{n}\right)-\psi\left(\frac{n-i}{n}\right)$ may be interpreted as the right derivative of ψ at $n-i/n$. The ψ' function assigns weights to the incomes in accordance with their rank in the income distribution.

Similarly, we can define $\psi^{p+1}\left(\frac{n-i}{n}\right) := \psi^{p}\left(\frac{n-i+1}{n}\right)-\psi^{p}\left(\frac{n-i}{n}\right) \quad with \quad p \ge 1$

Particular cases of social evaluation functions are the extended Gini social evaluation function defined as[10]

$$SG_{ini}(x) = \sum_{i=1}^{n}\left[\left(\frac{n-i+1}{n}\right)^{\theta}-\left(\frac{n-i}{n}\right)^{\theta}\right]x_i \text{ for } \theta > 1 \tag{3}$$

When $\theta = 2$ we have the Gini evaluation function,[11]

Following Mehran (1976) any social welfare function W_ψ defined as in eq.(2) satisfies PD if and only if $\psi''\left(\frac{n-i}{n}\right) > 0$ and satisfies PTS if and only if $\psi'''\left(\frac{n-i}{n}\right) \ge 0$. Therefore, for the family of relative extended Gini functions in eq (3); PD is satisfied if and only if $\theta > 1$; and PD and PTS if and only if $\theta \ge 2$.

Aaberge (2009) analyzes the implications of UPTS for any twice differentiable function W_ψ defined as in eq.(2) and shows that PD and UPTS are satisfied if and only if $\psi''\left(\frac{n-i}{n}\right) > 0$ and $\psi'''\left(\frac{n-i}{n}\right) \le 0$. It is straightforward to see that for the case of the relative extended Gini family in eq. (3) PD and UPTS are satisfied if and only if $1 \le \theta \le 2$.[12]

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[8] See among others Mehran (1976), Yaari (1988), Donaldson and Weymark (1980), Weymark (1981), Aaberge (2001, 2009), Chateauneuf and Moyes (2006), Moyes (2007) and Magdalou and Moyes (2009).

[9] The continuous version of eq. (2) is $W_v(x) := \int_0^1 v(\bar{t})F^{-1}(t)\,dt$. where $v(\bar{t}) = \psi'(\bar{t}) \quad with \quad \bar{t} = 1-t$

[10] See Donaldson and Weymark (1983), and Yitzhaki (1983).

[11] The continuous version of eq. (3) is given by $SG_{ini}(x) = \int_0^1 \theta\bar{t}^{\theta-1}F^{-1}(t)\,dt$ for $\theta > 1$, where $\bar{t} = 1-t$. When $\theta = 2$ we have the Gini function.

[12] Other measures fulfilling UPTS can be found in Aaberge (2001) and in Goerlich et al. (2009).

3. Rank dependent social evaluation functions and up-down positional transfer sensitivity

In this section we introduce a new principle that strengthens PD, up-down positional transfer sensitivity.[13] Its implications for rank dependent social evaluation functions are analyzed as well.

Definition 4. Up-down positional transfer sensitivity, UDPTS. A social evaluation function W is said to satisfy up-down positional transfer sensitivity, if for any pair of individuals with incomes $F^{-1}(i)$ and $F^{-1}(j)$, $\Delta W(i,\delta,\rho)\geq \Delta W(j,\delta,\rho)$ when $\frac{i+\rho}{n}<\frac{j+\rho}{n}\leq\frac{1}{2}$, and $\Delta W(i,\delta,\rho)\leq \Delta W(j,\delta,\rho)$ when $\frac{1}{2}\leq\frac{i}{n}<\frac{j}{n}$.

For any social evaluation function defined as in eq.(1) that satisfies UDPTS, the following proposition states that the weights should be convex below the median and concave above.

Proposition 1. *Any rank dependent social evaluation function W_ψ defined as in eq.(1) satisfies UDPTS if and only if $\psi''\left(\frac{n-i}{n}\right)\geq 0$ for all $\frac{i}{n}\leq\frac{1}{2}$ and $\psi''\left(\frac{n-i}{n}\right)\leq 0$ for all $\frac{i}{n}\geq\frac{1}{2}$.*

Proof: In the appendix.

Lambert and Lanza (2006), taking into account the concept of relative risk aversion in Pratt (1964), introduce a relative indicator of the lower tail concern for positional inequality measures. In our context, we take $Q_\psi(\bar{t})=\frac{\psi'''(\bar{t})}{\psi''(\bar{t})}\bar{t}$ where $\bar{t}=\frac{n-i}{n}$ as the relative indicator of the lower tail concern for positional social evaluation functions. When PTS is assumed and the measure exhibits downside inequality aversion, $Q_\psi(\bar{t})$ is positive and is determined by "the degree of convexity" of the weighting function. By contrast, if a measure fulfils UPTS, it exhibits upside inequality aversion and the weighting function is concave. In this case, the opposite of $Q_\psi(\bar{t})$ can be considered as an indicator of the strength of upper tail concern and depends on the level of concavity of the weighting function.

If we take into account a measure satisfying UDPTS, the level of convexity of the weighting function below the median, and that of the concavity above, this gives us an idea of the lower tail concern below the median and the upper tail concern above. Thus $\left|Q_\psi(\bar{t})\right|$ is an appropriate indicator for these measures. A link between this indicator and the new property is obtained in the following result.

Proposition 2. *Let W_ψ and $\hat{W}_\psi$ be two twice differentiable rank dependent social evaluation functions defined as in eq.(2) by respectively, ψ' and $\hat{\psi}'$, such that $\hat{\psi}'=T(\psi')$* for all $\bar{t}=\frac{n-i}{n}\geq\frac{1}{2}$, $\hat{\psi}'=S(\psi')$ for all $\bar{t}=\frac{n-i}{n}\leq\frac{1}{2}$, *and $T'(\ )>0, T''(\ )\geq 0$, $S'(\ )>0, S''(\ )\leq 0$.*

If W_ψ *satisfies UDPTS, then:*

- *i)* $\hat{W}_\psi$ *satisfies UDPTS*
- *ii)* $\left|Q_\psi(\bar{t})\right|\leq\left|Q_{\hat{\psi}}(\bar{t})\right|$ for all $\bar{t}\in[0,1]$.

[13] If we come back to the example in the Introduction section, up-down positional transfer sensitivity demands that $W(z^1)\geq W(z^2)$.

Proof: In the appendix.

4. Some rank dependent social evaluation functions fulfilling up-down positional transfer sensitivity

In this section we consider a particular case of social evaluation function defined by eq.(1) where the weighting function is given by $\psi\left(\frac{n-i}{n}\right)=\frac{1}{\theta}\left(\left(\frac{n-i}{n}\right)^{\theta-1}+\left(\frac{i}{n}\right)^{\theta-1}-1\right)+\frac{n-i}{n}$.

$$SG_\theta^{ud}\left(x\right)=\sum_{i=1}^{n}\left[\frac{1}{\theta}\left(\left(\frac{n-i+1}{n}\right)^{\theta}-\left(\frac{n-i}{n}\right)^{\theta}-\left(\left(\frac{i}{n}\right)^{\theta}-\left(\frac{i-1}{n}\right)^{\theta}\right)\right)+\frac{1}{n}\right]x_i \quad \text{for} \quad \begin{cases}1<\theta\le 2\\ \text{or}\\ \theta\ge 3\end{cases} \tag{4}$$

From proposition 1 it is straightforward to prove that this family fulfils UDPTS.[14]

For both parameter values, $\theta=2$ and $\theta=3$, the weights are linear and the function becomes the Gini social evaluation function.

Moreover, one characteristic of this class is that, for each income distribution, the social evaluation value SG_θ^{ud} varies continuously as a function of the θ-parameter. As mentioned above, an indicator of the lower tail concern below the median and the upper tail concern above is given by $\left|Q_\psi\left(\overline{t}\right)\right|$. For the family in eq.(4), this indicator depends on θ and becomes

$$\left|Q_\theta\left(\overline{t}\right)\right|=\left|\frac{\left(\theta-2\right)\left(\overline{t}^{\theta-3}-\left(1-\overline{t}\right)^{\theta-3}\right)}{\overline{t}^{\theta-2}+\left(1-\overline{t}\right)^{\theta-2}}\right|\overline{t} \tag{5}$$

Actually for each $t\in\left[0,1\right]$, and from $\theta=3$ when the weights are linear, as θ increases, this indicator also increases. Thus, the curvature of the weighting function becomes greater, and from proposition 1 it can be concluded that it turns out to be more convex below the median and more concave above. The limiting case is when $\theta\to\infty$, giving $SG_\theta^{ud}\left(x\right)\to\min_t F^{-1}\left(t\right)-\max_t F^{-1}\left(t\right)$, when the weights tends to zero except for the individuals at the extreme of the tails of the distribution. Therefore, the measure only considers transfers either from the richest individual, or to the poorest one.

Now, if we consider $\theta=2$, once again we get the Gini social evaluation function, and for each income distribution, as θ decreases $\left|Q_\theta\left(t\right)\right|$ increases, then the index becomes more sensitive to transfers at the tails of the distribution. The limiting case is when $\theta\to 1$, giving $SG_\theta^{ud}\left(x\right)\to\min_t F^{-1}\left(t\right)-\max_t F^{-1}\left(t\right)$. However, the $2<\theta<3$ indices in eq.(4) do not fulfil UDPTS. Actually they have a concave weighting function below the median and convex above. Thus they exhibit upside inequality aversion below the median and downside inequality aversion above, just the opposite to the requirements of UDPTS.

5. Conclusion

In this paper, a new property, UDPTS, which strengthens PD has been introduced. It is shown that the weighting function of any rank dependent social evaluation function fulfilling UDPTS should be convex below the median and concave above. Moreover, any other social evaluation function, with a more convex weighting function below the median and more concave above, should rank pairs of distributions in the same way. In this paper we provide a particular class of social evaluation functions fulfilling this property. The Gini social

[14] The continuous version of eq. (4) is $W_v\left(x\right):=\int_0^1\left(\overline{t}^{\theta-1}-\left(1-\overline{t}\right)^{\theta-1}+1\right)F^{-1}\left(t\right)dt$ for $\theta>1$, where $\overline{t}=1-t$.

evaluation function belongs to this class. The relative and absolute inequality indices, defined in the usual way, inherit this property, UDPTS, of the corresponding social evaluation function up to a change of sign.

Appendix

Proposition 1. *Any rank dependent social evaluation function* W_ψ *defined as in eq.(1) satisfies UDPTS if and only if* $\psi'''\left(\frac{n-i}{n}\right)\geq 0$ *for all* $\frac{i}{n}\leq\frac{1}{2}$ *and* $\psi'''\left(\frac{n-i}{n}\right)\leq 0$ *for all* $\frac{i}{n}\geq\frac{1}{2}$.

Proof: Given a small progressive transfer $\delta>0$ from an individual at the i^{th} percentile of the distribution to another individual at the $(i-\rho)^{th}$ percentile, from eq.(1) we obtain, $\Delta W\left(\frac{i}{n},\delta,\frac{\rho}{n}\right)=\psi'\left(\frac{n-(i-\rho)}{n}\right)\delta-\psi'\left(\frac{n-i}{n}\right)\delta$.

I satisfies UDPTS if and only if

$$\Delta W\left(\frac{i}{n},\delta,\frac{\rho}{n}\right)\geq\Delta W\left(\frac{j}{n},\delta,\frac{\rho}{n}\right)\Leftrightarrow$$

$$\psi'\left(\frac{n-(i-\rho)}{n}\right)\delta-\psi'\left(\frac{n-i}{n}\right)\delta\geq\psi'\left(\frac{n-(j-\rho)}{n}\right)\delta-\psi'\left(\frac{n-j}{n}\right)\delta \text{ for all } \frac{i}{n}<\frac{j}{n}\leq\frac{1}{2}, \text{ and}$$

$$\Delta W\left(\frac{i}{n},\delta,\frac{\rho}{n}\right)\leq\Delta W\left(\frac{j}{n},\delta,\frac{\rho}{n}\right)\Leftrightarrow$$

$$\psi'\left(\frac{n-(i-\rho)}{n}\right)\delta-\psi'\left(\frac{n-i}{n}\right)\delta\leq\psi'\left(\frac{n-(j-\rho)}{n}\right)\delta-\psi'\left(\frac{n-j}{n}\right)\delta \text{ for all } \frac{1}{2}\leq\frac{i-\rho}{n}<\frac{j-\rho}{n}.$$

Equivalently, *I* satisfies UDPTS if and only if $\psi'''\left(\frac{n-i}{n}\right)\geq 0$ for all $\frac{i}{n}\leq\frac{1}{2}$, and $\psi'''\left(\frac{n-i}{n}\right)\leq 0$ for all $\frac{1}{2}\leq\frac{i}{n}$. Q.E.D.

Proposition 2. *Let* W_ψ *and* $\hat{W}_\psi$ *be two twice differentiable rank dependent social evaluation functions defined as in eq.(2) by respectively,* ψ' *and* $\hat{\psi}'$, *such that* $\hat{\psi}'=T(\psi')$ for all $\bar{t}=\frac{n-i}{n}\geq\frac{1}{2}$, $\hat{\psi}'=S(\psi')$ for all $\bar{t}=\frac{n-i}{n}\leq\frac{1}{2}$, *and* $T'(\ )>0, T''(\ )\geq 0$, $S'(\ )>0, S''(\ )\leq 0$.

If W_ψ *satisfies UDPTS, then:*

i) $\hat{W}_\psi$ *satisfies UDPTS*

ii) $\left|Q_\psi(\bar{t})\right|\leq\left|Q_{\hat{\psi}}(\bar{t})\right|$ for all $\bar{t}\in[0,1]$.

Proof:

i) Clearly, $\widehat{\psi}''(\bar{t})=T'(\psi'(\bar{t}))\psi''(\bar{t})$ for all $\bar{t}=\frac{n-i}{n}\geq\frac{1}{2}$ and $\widehat{\psi}''(\bar{t})=S'(\psi'(\bar{t}))\psi''(\bar{t})$ for all $\bar{t}=\frac{n-i}{n}\leq\frac{1}{2}$. Since $\psi''(\bar{t})>0$, $T'(\ )>0$ and $S'(\ )>0$, we have that $\widehat{\psi}''(\bar{t})>0$ for all $\bar{t}$.

For all $\bar{t}=\frac{n-i}{n}\geq\frac{1}{2}$, $\widehat{\psi}'''(\bar{t})=T''(\psi'(\bar{t}))(\psi''(\bar{t}))^2+T'(\psi'(\bar{t}))\psi'''(\bar{t})$ and by proposition 1 we know that $\psi'''(\bar{t})\geq 0$. Thus, $\widehat{\psi}'''(\bar{t})$ for all $\bar{t}=\frac{n-i}{n}\geq\frac{1}{2}$.

For all $\bar{t}=\frac{n-i}{n}\leq\frac{1}{2}$, $\widehat{\psi}'''(\bar{t})=S''(\psi'(\bar{t}))(\psi''(\bar{t}))^2+S'(\psi'(\bar{t}))\psi'''(\bar{t})$ and by proposition 1 we know that $\psi'''(\bar{t})\leq 0$. Thus, $\widehat{\psi}'''(\bar{t})\leq 0$ for all $\bar{t}=\frac{n-i}{n}\leq\frac{1}{2}$, and we get the result.

ii) For all $\bar{t}=\frac{n-i}{n}\geq\frac{1}{2}$, we have that $\widehat{\psi}''(\bar{t})=T'(\psi'(\bar{t}))\psi''(\bar{t})$, then the following expression holds $\frac{d}{dt}\left[\ln T'(\psi'(\bar{t}))\right]=\frac{d}{dt}\left[\ln\frac{\widehat{\psi}''(\bar{t})}{\psi''(\bar{t})}\right]$ and it gives

$\frac{T''(\psi'(\bar{t}))}{T'(\psi'(\bar{t}))}\psi''(\bar{t})=\frac{\widehat{\psi}'''(\bar{t})}{\widehat{\psi}''(\bar{t})}-\frac{\psi'''(\bar{t})}{\psi''(\bar{t})}$. Hence, given $\psi''(\bar{t})>0$, $T'(\quad)>0$, and $T''\geq 0$, it holds that $\frac{\widehat{\psi}'''(\bar{t})}{\widehat{\psi}''(\bar{t})}\geq\frac{\psi'''(\bar{t})}{\psi''(\bar{t})}$.

For all $\bar{t}=\frac{n-i}{n}\leq\frac{1}{2}$, considering $\widehat{\psi}'=S(\psi')$ and $\widehat{\psi}''=S'(\psi')\psi''$, we may apply a similar reasoning to get the result. Q.E.D.

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