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Provision of a discrete public good with infinitely-many commodities

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Abstract

Suppose a group of individuals must decide whether to undertake a public project. The private commodity space, from which are also drawn the inputs for the public good, exhibits the Riesz decomposition property. We give a sufficient condition for the existence of a feasible provision of the public good that Pareto-dominates inaction. The condition is that the `net benefit' from the public project be positive. If this condition is met, by the Riesz decomposition property the cost of the project can be decomposed into a sum of individual contributions or taxes so that the project can be `financed' and every agent retains a positive surplus.

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1. Introduction

This paper contributes to the literature on economies with public goods and infinitely-many private commodities. We develop a simple model to analyze the conditions under which undertaking a public project Pareto-dominates not providing the public good. By using the Riesz decomposition property, we obtain a condition that is reminiscent of the standard cost-benefit analysis in quasilinear finite-dimensional environments: a measure of the net benefit of the project should be positive.

The Riesz decomposition property has played a key role in the analysis of existence and optimality of equilibrium allocations in infinite-dimensional economies with only private goods. We show that it is also useful for the analysis of economies with public goods and that it fosters geometric intuition in a non-euclidean setting. The Riesz decomposition property ensures that if the sum of each individual's 'willingness-to-pay' for a public project exceeds the 'cost' of provision (in a sense which we make precise in section 5), then there exists a profile of individual contributions (or taxes) that raise enough revenue to finance the cost of the public good and that generate a positive surplus for everybody. Clearly, the Riesz decomposition property is a generalization of a proposition that holds true in any euclidean space. Therefore, its application in more general settings can be geometrically appealing.

To understand the thrust of our contribution, first consider the canonical setting in which there are only two goods: a pure public good available in discrete amounts and a private good which can be thought of as money. In this scenario, recall that providing the public good Pareto dominates not providing it, if and only if the sum of consumers willingness-to-pay is greater than the cost of provision of the public good.[1] In our opinion, this simple necessary and sufficient condition is noteworthy because it provides a theoretical underpinning for the cost-benefits analysis approach (provided that the consumer's benefit from the public good is measured by the willingness-to-pay).[2]

It is widely acknowledged, however, that there are many contexts in which infinite dimensional models arise naturally and must be used to address real policy questions (dynamic models, uncertainty, commodity differentiation and so forth). Therefore, our goal in this study is to extend the afore-said sufficient condition to an infinite-dimensional economy. More precisely, we consider a (pure) discrete public good and infinitely-many private commodities that can be employed as inputs to produce the public good. Clearly, private commodities are used also for consumption purposes. The production technology is such that one can easily define the notion of 'cost' of provision of the public good. In short, the cost is the 'minimum' amount of private goods required to produce the public good. Clearly, production involves giving up consumption. To

[1] See, for instance, chapter 23 in Varian (1992).

[2] The willingness-to-pay is not observable and typically is not truthfully-revealed. Addressing these issues is outside the scope of this paper. For an interesting study of testable restrictions of Pareto optimality in an economy with public goods, see Snyder (1999).

carry out such extension, we assume that the private commodity space is a Riesz space, which is a specific type of ordered vector space. Ordered vector spaces are important in economic analysis because, among other things, they provide an ordering on commodity vectors for which “more is better” according to consumers preferences. Furthermore, most spaces used in economic analysis are indeed Riesz spaces.[3] Most importantly, Riesz spaces satisfy a remarkable property known as the Riesz decomposition property.[4] Quoting Aliprantis *et al.* (2006): ‘Mas-Colell has suggested the following economic interpretation of the Riesz decomposition property. Interpret each $x_i \geq 0$ as the vector of holdings of person i. Then $x = \sum_{i=1}^{n} x_i$ represents the total wealth of the economy. Think of the vector y $(0 \leq y \leq x)$ as a tax. The Riesz Decomposition Property says that in a Riesz space, if the tax is feasible in the aggregate, then there is a feasible way to distribute the tax among the individuals.’

The above quotation did provide the stimulus to our analysis. Imagine framing Mas-Colell’s remark in a framework in which each member of a certain community might be “taxed” (in terms of forgone consumption goods) to cover the ‘cost’ of a public project. We shall define the notion of willingness-to-pay for the project in this setting, and then we shall prove that it is well-defined. Now, with the above Mas-Colell’s observation in mind, intuition suggests that, by using somehow the Riesz decomposition property, it should be possible to find a feasible distribution of the ‘cost’ among the individuals that makes everyone better-off. As a matter of fact, under the hypothesis that the sum of agents’ willingness-to-pay ‘exceeds’ the ‘cost’ of provision of the public good, in this paper we do substantiate the foregoing intuition.

The lay-out of the paper is as follows: in section 2 we discuss briefly the most closely-related literature of which we are aware. In section 3 we describe the economic environment. In section 4 we spell out our assumptions and the rationale for them. In section 5 we first present an instrumental lemma, and then we state and prove our main result.

2. Related literature

Khan *et al.* (1988) prove a version of the second welfare theorem in non-convex economies with public goods. They assume that the commodity space is an ordered locally-convex space. Likewise, we do not assume convexity of preferences and the production set. However, our private commodity space is a fairly general ordered topological vector space that subsumes locally-convex spaces (see section 4). De Simone *et al.* (2004) establish versions of the two welfare theorems in production economies with public goods and an infinite-dimensional commodity space. In contrast, we do not examine implementation mechanisms. Moreover, we only consider one discrete public good whereas the authors work

[3] See, e.g., Aliprantis *et al.* (1983).

[4] For more on the Riesz decomposition property, see section 4 and the references therein.

with an abstract set of public projects. Del Mercato *et al.* (2006) introduce the notion of Edgeworth equilibrium for an economy with public goods and infinitely-many private goods. They show that Edgeworth equilibria exist and can be decentralized as Lindahl–Foley equilibria. In contrast, we only study efficiency with a discrete public good, and we are not concerned with decentralization. Graziano (2007) deals with the fundamental theorems of welfare economics in economies with infinitely many private goods and a set of public projects. She makes fairly general assumptions on the fundamentals of the economy. Her private commodity space need not satisfy the Riesz decomposition property, whereas the commodity space we introduce does satisfy that property. Moreover, in Graziano (2007) agents are endowed with a non-ordered preference relation. In contrast, we work with ordered preferences because for our purposes it is important that reservation prices be well-defined and uniquely determined (see section 5). Habte *et al.* (2011) study non-convex economies with public goods and infinite-dimensional commodity spaces. They focus on generalizations of the second welfare theorem. Their research program is quite remarkable. We do not study how an efficient allocation with public goods may be implemented. On the other hand, Habte *et al.* (2011) employ tools based on generalized differentiation and this prompts them to impose specific restrictions on the commodity space. In contrast, our result hinges upon the Riesz decomposition property, and therefore we can allow for fairly general private commodity spaces (see section 4).

3. The economic environment

We consider an economy with a finite set of agents H, an infinite-dimensional private commodity space L, and private provision of one pure public good. For our simple purposes, the economy can be formalized as follows:

$$\xi = \left(\mathbb{R} \times L,\ \ (X_h, \succeq_h, e_h)_{h \in H}\, , Y\right)$$

L is an ordered vector space, and clearly $\mathbb{R} \times L$ is the commodity space. For each agent $h \in H$, $X_h = \mathbb{R}_+ \times L_+$ is the consumption set, where L_+ is the positive cone of L. $\mathbb{R}$ is equipped with the euclidean topology, and L is assumed to be equipped with a linear Hausdorff topology τ. Consequently, $\mathbb{R} \times L$ is endowed with the product topology. Each agent h is endowed with $e_h = (0, \omega_h) \in \mathbb{R}_+ \times L_+$, with $\omega_h \neq 0$ (there is no initial endowment of the public good). Each agent h is endowed with a preference relation $\succeq_h$defined on $\mathbb{R}_+ \times L_+$. This preference relation is assumed to be a preference order: it is reflexive, complete, and transitive. Private goods are both consumption goods and inputs for the production of the public good. Specifically, each agent h initially has some endowment of the private goods, ω_h, and can contribute to the production of the public good. If individual h contributes $g_h \in L_+$, with $\omega_h - g_h \in L_+$, then she has $x_h = \omega_h - g_h$ of private goods consumption. We assume that the pure public good is only available in discrete amount; either it is provided in that amount (say 1), or it is not provided at all. The interpretation of the amount of the public good being equal to 1 is that a certain public project is undertaken by the agents. $Y \subset \mathbb{R} \times L$ is the production set. Its elements are the feasible production plans necessary to transform the private goods into the public good. Next, we posit more specific assumptions that will be useful for the analysis.

4. Assumptions and discussion

A very good reference for the mathematical background employed in the sequel is Aliprantis *et al.* (2007).

Assumption 4.1. *L is a Riesz space.*

Remark 4.1 Because L is a Riesz space, it has the Riesz decomposition property.[5] We assume that L is a Riesz space because the Riesz decomposition property plays a central role in the proof of Proposition 5.1 below. A couple of comments are in order. First, the above assumption is fairly mild, as many familiar spaces of interest in Economics are Riesz spaces. Secondly, recall that an ordered vector space with the Riesz decomposition property need not be a Riesz space.[6] Hence, because ultimately we are interested in the Riesz decomposition property, the assumption that L is a Riesz space may be relaxed.

Assumption 4.2. *For each $h \in H$, $\succeq_h$is a continuous and strictly-monotonic preference relation. Furthermore, $(0, \omega_h) \succ_h (1, 0)$.*

Remark 4.2. It seems reasonable to assume that, if private and public consumption were mutually exclusive, agents would choose private consumption over the public good.

Assumption 4.3. *$(0, 0) \in Y$, and there exists a $c \in L_+$ such that $(1, -x) \in Y$ if and only if $x - c \in L_+$.*

Remark 4.3. The above assumption is a generalization of the standard textbook hypothesis adopted in the finite-dimensional setting.[7] It may be justified by assuming a fixed-coefficient technology with a single activity. The latter is either not activated at all (there is no public project), or it is activated at the unit level (the public project is undertaken).

5. Pareto-improving and feasible provision of the public good

[5] See, e.g., Aliprantis *et al.* (2007).
[6] See Examples 1.56 through 1.58 in Aliprantis *et al.* (2007).
[7] See, for example, Varian (1992).

In this section we provide an answer to the following question: when is the provision of the public good a Pareto-improvement upon no provision at all? The road-map is as follows: first we define the notion of reservation price in our infinite-dimensional setting, and we prove that it is well-defined. Basically, the reservation price is the maximum amount of the private goods the agent is willing to forgo to enjoy the public good (willingness-to-pay). Then, we state a sufficient condition for the provision of the public good to be Pareto-improving. The sufficient condition will be expressed in terms of the agents' reservation prices. The proof of it will hinge on the Riesz decomposition property.

Definition 5.1. *For each agent $h \in H$, $r_h \in L_+$ is said to be agent h reservation price if it satisfies the following condition:* $(1, \omega_h - r_h) \sim_h (0, \omega_h)$.

In what follows we prove that reservation prices exist and are uniquely determined. Our proof is inspired by Mas-Colell (1986) and the Monotone Representation Theorem in Becker and Boyd (1997). The intuition for Lemma 5.1 is straightforward: loosely put, the properties of the commodity space and preferences guarantee that the indifference surface through the bundle $(0, \omega_h)$ crosses the 'interior' of the segment joining $(1, 0)$ and $(1, \omega_h)$ only once.

Lemma 5.1. *Under Assumption 4.2, for each $h \in H$ there exists a unique reservation price.*

Proof: Pick any $h \in H$, and define the set

$$J_h = \{\theta\,(1,0) + (1-\theta)\,(1,\omega_h) : 0 \leq \theta \leq 1\}\,.$$

By completeness, $J_h = A_h \cup B_h$, where $A_h = \{y \in J_h : y \succeq_h (0, \omega_h)\}$ and $B_h = \{y \in J_h : y \preceq_h (0, \omega_h)\}$. By Assumption 4.2, A_h and B_h are both non-empty and closed in J_h. Because the given product topology is a linear topology on $\mathbb{R} \times L$, clearly J_h is connected. Therefore, $A_h \cap B_h \neq \varnothing$. That is, there exists a $0 \leq \theta_h \leq 1$ such that $(1, \omega_h - \theta_h\omega_h) \sim_h (0, \omega_h)$. Finally, put $\theta_h\omega_h = r_h$ and notice that $(1, \omega_h - r_h) \sim_h (0, \omega_h)$, with $r_h \in L_+$ and $\omega_h - r_h \in L_+$. As for uniqueness of r_h, it is very easy to check that strict monotonicity and transitivity of preferences imply that $A_h \cap B_h$ is a singleton.■

The proposition below is our main result. The Riesz decomposition property confers a neat geometric intuition on the proof: if the total 'benefit' from the public good is greater than the cost, then one can 'move down' from the vector of agents reservation prices onto the 'isocost' hyperplane in such a way that each agent's 'personalized cost' is less than her reservation price.

Proposition 5.1. *Under Assumptions* 4.1, 4.2, *and* 4.3, *if* $\sum_{h\in H} r_h - c \in L_+ \setminus \{0\}$, *then providing the public good Pareto dominates not providing it.*

Proof: Given Assumption 4.3, we must show that one can find a pattern of private contributions $(g_h)_{h\in H} \subset L_+$ such that $\sum_{h\in H} g_h - c \in L_+$ and $(1, \omega_h - g_h) \succeq_h (0, \omega_h)$ for all $h \in H$, with $(1, \omega_i - g_i) \succ_i (0, \omega_i)$ for at least one $i \in H$. To this end, notice that by the Riesz decomposition property,[8]

[8]See section 1.8 in Aliprantis *et al.* (2007), or section 8.5 in Aliprantis *et al.* (2006).

$\sum_{h\in H} r_h - c \in L_+\setminus\{0\}$ implies the existence of $(g_h)_{h\in H} \subset L_+$ such that $r_h - g_h \in L_+$ for all $h \in H$, and $c = \sum_{h\in H} g_h$. By assumption, $\sum_{h\in H} r_h - c \neq 0$. Therefore, $c = \sum_{h\in H} g_h$ implies that there is a $i \in H$ such that $g_i \neq r_i$. It then follows from transitivity and strict monotonicity of preferences, and from Definition 4.1, that $(1, \omega_h - g_h) \succeq_h (0, \omega_h)$ for all $h \in H$, and $(1, \omega_i - g_i) \succ_i (0, \omega_i)$. The proof is finished.■

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