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A characterization of a solution based on prices for a discrete cost sharing problem

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Abstract

In this paper we characterize a solution for a discrete cost sharing problem where each agent requires a set of services but they can share them with each other. Also, the services provider can offer any subset of services. So, the agents have to distribute the cost of using the whole set of services amongst them. In our proposed solution, we work with the cost function as a cooperative game, and the Shapley value of this game is regarded as the price by each service. Then, we divide equally the service price among the agents who consume it. In addition, we show some properties that our solution satisfies.

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1. Introduction

The general cost sharing problem considers a set of agents N who need to consume different quantities of m different goods M. Then, each agent $i \in N$ has a *demand vector* $M_i \in \mathbb{R}^m$ where the j-th entry corresponds to his demand of item $j \in M$. In addition, there exists a *cost function*, that assigns a real number to each subset of items that could be required. The problem is to distribute the cost, according to the cost function, of the total set of goods demanded by the agents. Under this general framework, there exist several studies of the problem: see Moulin (1995), Samet and Tauman (1982) and Wang and Zhu (2002).

The problem that we study in this paper is a simplified version of the general cost sharing problem. We regard the goods as discrete services and that each agent could require at most one unit of each service. The services are non-rivals and then, several agents may require the same service. Also, we consider that agents share the cost of common services. So, the cost function c is defined in subsets of M. We suppose that there exist at least one agent requiring each service. So, the problem is to find how to allocate the total cost, $c(M)$, among the agents. The general discrete cost sharing problem has been studied by several authors such as Calvo and Santos (2000) and Sprumont (2005).

Let us consider the next situation: A set of expositors needs the service of a security company for the surveillance of their equipment and personal. The expositors will be in the same physical space but not all of them all the time. There can be periods of time when some expositors share the space and then, it is more reasonable to share the costs of the surveillance service of that time. The security company offers its services without considering the number of persons or equipment they need to supervise. So, we propose a method for allocating the cost of the surveillance service for the total time that is required among the expositors.

2. Notation and definitions

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents and $Q = \{1, 2, \ldots, q\}$ the finite set of services. Each agent $i \in N$ needs some services in Q and these requirements are denoted by $M_i \subseteq Q$. As the services are considered non-rivals, it is possible that several agents require the same service. Notice that it is possible that an agent i does not need any service, and therefore, $M_i = \emptyset$. We denote $M = (M_1, M_2, \ldots, M_n)$ the *vector of services* required by the agents. Also, we define $P(Q)$, the power set of Q. We define a *cost function* $c : P(Q) \to \mathbb{R}$ with $c(\emptyset) = 0$. This function indicates, for a subset of services $S \subseteq Q$, that the cost related to these services is $c(S)$. For a set of services Q, the set of cost functions is denoted by $G_Q = \{c : P(Q) \to \mathbb{R} \mid c(\emptyset) = 0\}$. Also, it is easy to check that the set of cost functions for Q is a real vector space with the sum and scalar product defined in the usual way: for $c_1, c_2 \in G_Q$ and $\lambda \in \mathbb{R}$, $(c_1 + c_2)(S) := c_1(S) + c_2(S)$ and $(\lambda c_1)(S) := \lambda c_1(S)$ for all $S \in P(Q)$. We are interested in situations where there exists at least one agent that requires each service. That condition restricts the set of possible service vectors. Thus, given a set of services Q we define

$$E(Q) = \{(M_1, \ldots, M_n) \in P(Q) \times \cdots \times P(Q) \mid \forall j \in Q, \exists i \in N \text{ with } j \in M_i\},$$

the subset of service vectors that we are interested in. Then, a *discrete service problem*, or for simplicity, a *problem*, is defined as a pair (c, M), $c \in G_Q$ and $M \in E(Q)$. Henceforth, we consider that the set of agents N and the set of services Q are given and fixed.

For a vector of services $M \in E(Q)$, we define, for each $S \subseteq N$ and $T \subseteq Q$

$$S_T(M) = \{i \in S \mid \exists j \in T \text{ such that } j \in M_i\}.$$

This set represents the subset of agents in S that requires some service in T according to M. In the same way, we define, for each $S \subseteq N$, $M_S = \cup_{i \in S} M_i$. This set indicates the subset of services that is required for some agent in S.

With the above terminology, we define a solution as an operator

$$\varphi : G_Q \times E(Q) \to \mathbb{R}^n.$$

For a given problem (c, M), a solution $\varphi_i(c, M)$ indicates the amount that agent i must pay for his required services according to the cost function c and the vector of requirements M.

Example 1. *A surveillance and security company offers its services from 8:00 a.m. to 8:00 p.m. They divide their workday in four blocks: Block 1 is from 8:00 a.m. to 11:00 a.m.; Block 2 is from 11:00 a.m. to 2:00 p.m.; the third block is from 2:00 p.m. to 5:00 p.m. and block 4 is from 5:00 p.m. to 8:00 p.m. The related cost function is*

$$\begin{array}{rrr} c(\{1\}) = 10 & c(\{1,3\}) = 30 & c(\{1,2,3\}) = 32 \\ c(\{2\}) = 10 & c(\{1,4\}) = 40 & c(\{1,2,4\}) = 48 \\ c(\{3\}) = 20 & c(\{2,3\}) = 26 & c(\{1,3,4\}) = 52 \\ c(\{4\}) = 30 & c(\{2,4\}) = 40 & c(\{2,3,4\}) = 52 \\ c(\{1,2\}) = 18 & c(\{3,4\}) = 44 & c(\{1,2,3,4\}) = 60. \end{array}$$

There are three persons who need the services of the security company for the protection of some equipment. Person 1 needs the services from 8:00 a.m. to 2:00 p.m. and from 5:00 p.m. to 8:00 p.m.; person 2 from 8:00 a.m. to 11:00 a.m. and from 2:00 p.m. to 5:00 p.m., and person 3 from 8:00 a.m. to 11:00 a.m. and from 2:00 p.m. to 8:00 p.m. We can write their requirements in terms of the working blocks offered by the security company as follows:

$$M_1 = \{1, 2, 4\} \quad M_2 = \{1, 3\} \quad M_3 = \{1, 3, 4\}.$$

According to this example, it is more convenient for the group of persons to join and hire the services of the security company from 8:00 to 20:00 rather than hire them individually. We have all the ingredients of a discrete service problem. A solution for this problem indicates how much each person must pay for the hired services.

In addition, we present some basic definitions about cooperative game theory. A cooperative game is defined as a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : P(N) \to \mathbb{R}$, with $v(\emptyset) = 0$, is the *characteristic function*. Given $S \subseteq N$, $v(S)$ is interpreted as the worth that agents in S can obtain because of their cooperation. There are several characterized solutions for a cooperative game; one of the most studied and accepted is the well-known *Shapley value*, given by the next formulation:

$$\mathrm{Sh}_i(N, v) = \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!} \left[v(S) - v(S \backslash \{i\}) \right].$$

For more details about the meaning and the properties of the Shapley value, please refer to Shapley (1953).

3. Characterization

In this section we work with an axiomatic approach for finding a solution for a discrete services problem.

Axiom 1. *(Additivity) Given two problems (c_1, M) and (c_2, M), a solution φ satisfies the additivity axiom if*

$$\varphi(c_1 + c_2, M) = \varphi(c_1, M) + \varphi(c_2, M).$$

This axiom is a very well-known property for cost sharing problems.

We are going to suppose that the services are hired in a unique block: the whole set of services. This make sense when the cost of hiring a set of services is not higher than the sum of the costs of hiring them individually (or in a partition of the set). This is, under our consideration, the predominant behavior in the majority of real life situations.

Axiom 2. *(Efficiency) Given a problem (c, M), a solution φ satisfies the efficiency axiom if*

$$\sum_{i \in N} \varphi_i(c, M) = c(Q).$$

According to an efficient solution, the cost of the whole set of services must be totally allocated among the agents.

For a requirements vector $M \in E(Q)$, we say that an agent $i \in N$ is *dummy* in M if $M_i = \emptyset$.

Axiom 3. *(Dummy agent) For a problem (c, M), a solution φ satisfies the dummy agent axiom if*

$$\varphi_i(c, M) = 0$$

for every dummy agent $i \in N$ in M.

In a dummy solution, the agent who does not have any requirements does not need to make a payment.

For a problem $(c, M) \in G_Q \times E(Q)$, a set of services $T \subset Q$ is *free* in (c, M) if

$$c(S) = 0 \quad \forall S \subseteq T \text{ and } c(S \cup P) = c(P) \quad \forall S \subseteq T \text{ and } \forall P \subseteq Q \backslash T.$$

Also, given a problem $(c, M) \in G_Q \times E(Q)$ and a subset of services $S \subseteq Q$, we define a new problem $(c - S, M - S) \in G_{Q \backslash S} \times E(Q \backslash S)$, that we call *the S-reduced problem*, where the new cost function and the new requirements vector are defined as follows:

$$(c - S)(T) = c(T) \quad \forall T \subseteq Q \backslash S \text{ and } (M - S)_i = M_i \cap (Q \backslash S) \qquad \forall i \in N.$$

In a S-reduced problem, the services in S are removed from M.

Axiom 4. *(Reduction) For a problem* (c, M)*, a solution* φ *satisfies the reduction axiom if, for each set of free services* $T \subseteq Q$ *in* (c, M)

$$\varphi(c, M) = \varphi(c - T, M - T).$$

For a vector of requirements $M \in E(Q)$, we say that two agents $i, j \in N$, are *equivalents* in (c, M) if $M_i = M_j$.

Axiom 5. *(Equivalent agents) Given a problem* (c, M)*, a solution* φ *satisfies the equivalent agents axiom if*

$$\varphi_i(c, M) = \varphi_j(c, M)$$

for each pair of equivalent agents $i, j \in N$ *in* (c, M)*.*

Two agents are equivalent if their requirements are the same. So, a solution which satisfies the previous axiom indicates that agents with the same requirements have to pay the same amount for their services.

In the same direction, for a cost function $c \in G_Q$, we say that two services $h, k \in Q$, $h \neq k$, are *substitutes* in c if for every $S \subseteq Q \backslash \{h, k\}$ we have $c(S \cup \{h\}) = c(S \cup \{k\})$. In addition, we say a problem (c, M) is *substitute* if all the services are substitutes in c.

Axiom 6. *(Substitute trivial problem) A solution* φ *satisfies the substitute trivial problem axiom if*

$$\varphi_i(c, M) = 0 \qquad \forall i \in N$$

when (c, M) *is a substitute problem with* $c(Q) = 0$*.*

In a substitute problem it is not possible to distinguish among the importance of the services. Then, we can consider that each service contributes to the total cost in the same way. The previous axiom establishes that if the cost of the whole set of services is equal to zero, and they are substitutes, the payment corresponding to each agent must be equal to zero also.

Theorem 1. *There exists a unique solution* $\varphi : G_Q \times E(Q)$ *that satisfies additivity, efficiency, dummy agents, reduction, equivalent agents and substitute trivial problem axioms. Even more, that solution has the next formulation:*

$$\varphi_i(c, M) = \sum_{j \in M_i} \frac{Sh_j(Q, c)}{|N_j(M)|} \qquad \forall i \in N. \tag{1}$$

Proof. First, we prove the uniqueness of the solution. Defining $\delta_\emptyset = 0$ and $\delta_S = c(S) - \sum_{T \subsetneq S} \delta_T$, we can write any cost function $c \in G_Q$ in the form

$$c = \sum_{S \subseteq N} \delta_S c_S$$

where $c_S \in G_Q$ is defined as $c_S(T) = 1$ if $S \subseteq T$ and zero otherwise. These cost functions are commonly known as *unanimity cost functions.* Let φ be a solution that satisfies the axioms of the theorem. Because of the additivity axiom, we have

$$\varphi_i(c, M) = \sum_{S \subseteq N} \varphi_i(\delta_S c_S, M).$$

So, it is sufficient to prove the uniqueness of the solution in each unanimity cost function. Given $S \subseteq Q$, the services in $Q \backslash S$ are free in c_S. So, because of the reduction axiom, the next condition holds

$$\varphi_i(\delta_S c_S, M) = \varphi_i(\delta_S c_S - Q\backslash S, M - Q\backslash S).$$

Now, we define the cost functions $c_r^j, \widehat{c}_r \in G_Q$, for all $j \in Q, r \in \mathbb{R}$:

$$c_r^j(T) = \begin{cases} r, & \text{if } j \in T; \\ 0, & \text{otherwise,} \end{cases} \qquad \widehat{c}_r(T) = \begin{cases} |T|r, & \text{if } T \neq N; \\ 0, & \text{otherwise,} \end{cases}$$

for all $T \subseteq Q$. So, we have the next decomposition:

$$\delta_S c_S - Q\backslash S = \sum_{j \in S} \delta_S c^j_{1/|S|} - \delta_S \widehat{c}_{1/|S|}$$

with $c^j_{1/|S|}, \widehat{c}_{1/|S|} \in G_S$ for all $j \in S$. By the additivity axiom we have

$$\varphi_i(\delta_S c_S - Q\backslash S, M - Q\backslash S) = \sum_{j \in S} \varphi_i(\delta_S c^j_{1/|S|}, M - Q\backslash S) - \varphi_i(\delta_S \widehat{c}_{1/|S|}, M - Q\backslash S).$$

Notice that $(\delta_S \widehat{c}_{1/|S|}, M - Q\backslash S)$ is a substitute problem. Because φ satisfies the substitute trivial axiom we have

$$\varphi_i(\delta_S \widehat{c}_{1/|S|}, M - Q\backslash S) = 0 \qquad \forall i \in N.$$

Additionally, in each problem $(\delta_S c^j_{1/|S|}, M - Q\backslash S)$ the services $S\backslash\{j\}$ are free. Due to the reduction axiom, we have:

$$\varphi_i(\delta_S c_S - Q\backslash S, M - Q\backslash S) = \sum_{j \in S} \varphi_i(\delta_S c^j_{1/|S|} - Q\backslash\{j\}, M - Q\backslash\{j\}).$$

It is important to notice that every problem $(\delta_S c^j_{1/|S|} - S\backslash\{j\}, M - Q\backslash\{j\})$ consists of a discrete service problem with only one service (the service j). In this specific kind of problems there are two kinds of agents: those who require the service and those who do not. Because of the dummy agents, equivalent agents and efficiency axioms, we have

$$\varphi_i(\delta_S c^j_{1/|S|} - Q\backslash\{j\}, M - Q\backslash\{j\}) = \begin{cases} 0, & \text{if } j \notin M_i; \\ \dfrac{\delta_S}{|S| \cdot |N_j(M)|}, & \text{otherwise.} \end{cases}$$

Thus, we have determined the solution uniquely in each reduced game. This proves the uniqueness of ϕ.

We need to prove that (1) satisfies the properties given in the theorem. Since the Shapley value is an additive value in cooperative games, the proof that (1) is an additive solution is trivial. Because of the efficiency of the Shapley value, we have

$$\sum_{i \in N} \varphi_i(c, M) = \sum_{i \in N} \left\{ \sum_{j \in M_i} \frac{\mathrm{Sh}_j(Q, c)}{|N_j(M)|} \right\} = \sum_{j \in Q} \frac{|N_j(M)| \, (\mathrm{Sh}_j(Q, c))}{|N_j(M)|} = c(Q).$$

Then, (1) is an efficient solution. For a dummy agent $i \in N$ in M we have $M_i = \emptyset$. Then, trivially, (1) satisfies the dummy agent axiom. In the same way, it is very easy to verify that (1) satisfies the equivalent agent axiom.

Let $T \subseteq Q$ be a set of free services in (c, M), then

$$\varphi_i(c, M) = \sum_{j \in (M_i \setminus T)} \frac{\mathrm{Sh}_j(Q, c)}{|N_j(M)|} + \sum_{j \in (M_i \cap T)} \frac{\mathrm{Sh}_j(Q, c)}{|N_j(M)|}$$

and because $\mathrm{Sh}_j(Q, c) = 0$ if $j \in T$, then

$$\varphi_i(c, M) = \sum_{j \in (M_i \setminus T)} \frac{\mathrm{Sh}_j(Q, c)}{|N_j(M)|}.$$

On the other hand,

$$\varphi_i(c - T, M - T) = \sum_{j \in (M_i \setminus T)} \frac{\mathrm{Sh}_j(Q \setminus T, c - T)}{|N_j(M)|} = \sum_{j \in (M_i \setminus T)} \frac{\mathrm{Sh}_j(Q, c)}{|N_j(M)|}.$$

This verifies that (1) satisfies the reduction axiom.

Let (c, M) be a substitute problem with $c(Q) = 0$. As $\mathrm{Sh}_j(Q, c) = 0$ for every $j \in Q$, (1) satisfies the substitute trivial problem axiom. Then, the proof ends. $\square$

According to solution (1), as a first step, we calculate the price of each service considering the cost function c as a cooperative game, and we calculate its Shapley value. After that, we make an equalitarian allocation of the price of each service among the agents who require it. Adding up each service of his set of demands, the amount that agent i must pay is given in (1).

Let us consider the situation given in Example 1. Considering the cost function as a cooperative game (Q, c), we obtain:

$$\mathrm{Sh}(Q, c) = (26/3, 8, 46/3, 28).$$

The previous solution indicates the real price of each block of hours, according to the Shapley value, if all the services are hired. Notice that the cost is less for each block, in contrast to hiring the block individually. Calculating (1) for this example:

$$\varphi(c, M) = (224/9, 95/9, 221/9).$$

Again, notice that there are savings due to the cooperation among the agents. If they do not cooperate, the individual costs are higher.

4. Some additional properties

Now, we are going to study additional properties of a solution ψ for a discrete cost sharing problem $(c, M) \in G_Q \times E(Q)$.

We say a cost function c is *non-decreasing* if $A \subseteq B \subseteq Q$ implies $c(A) \leq c(B)$.

Property 1. *(Monotony) A solution $\psi : G_Q \times E(Q) \to \mathbb{R}^n$ is monotone if for $M_k \subseteq M_l$ then $\psi_k(c, M) \leq \psi_l(c, M)$ for every non-decreasing cost function c.*

This property says that if agent l requires at least the same services as agent k, then the payoff for agent l is at least the payoff of agent k for non-decreasing cost functions.

Property 2. *(Reshuffling) For $S \subseteq N$ and $M, M' \in E(Q)$ such that $|S_j(M)| = |S_j(M')|$ for every $j \in Q$. A solution $\psi : G_Q \times E(Q) \to \mathbb{R}^n$ satisfies the reshuffling property if*

$$\sum_{i \in S} \psi_i(c, M) = \sum_{i \in S} \psi_i(c, M').$$

According to this property, if the agents in S interchange their services in some way, then the total payment of the group S does not change.

We say a cost function is *concave* if $c(S) + c(T) - c(S \cap T) \geq c(S \cup T)$ for all $S, T \in Q$. So, in a concave cost function it is profitable to hire the whole set of services.

Property 3. *(Individual rationality) A solution $\psi : G_Q \times E(Q) \to \mathbb{R}^n$ is individually rational if $\psi_i(c, M) \leq c(M_i)$ for all $i \in N$ when c is a concave cost function.*

In a concave cost function, the payoff of no one agent is punished by the cooperation.

Given $M \in E(Q)$, $h \in N$ and $k \in Q$, we define $(M + h_k) \in E(Q)$ as follows:

$$(M + h_k)_i = M_i \quad \text{if } i \in N \backslash \{h\} \text{ and } (M + h_k)_h = M_h \cup \{k\}.$$

Property 4. *(Independence of increasing demand) A solution $\psi : G_Q \times E(Q) \to \mathbb{R}^n$ is independent of increasing demand if*

$$\psi_i(c, M) = \psi_i(c, M + h_k) \qquad \textit{if } k \notin M_i, \qquad \forall i \in N.$$

Proposition 1. *The solution $\varphi : G_Q \times E(Q)$ given in (1) satisfies the (a) monotony, (b) reshuffling, (c) individual rationality and (d) independence of increasing demand properties.*

Proof. *(a)* The Shapley value of a problem (Q, c) where c is a non-decreasing cost function is always a non-negative number. The result easily follows from this fact. *(b)* The Shapley value of the services in the problems (c, M) and (c, M') is the same. Even more, $|N_j(M)| = |N_j(M')|$ for every $j \in M$. Then, the payoff for a *non-reshuffled* agent is the same in (c, M) and (c, M'). By efficiency of (1), the result follows. *(c)* The result is a consequence of properties of Shapley value and the core of concave cooperative games. For further information, refer to Shapley (1971). *(d)* The Shapley value of (Q, c) is the same in the problem (c, M) and $(c, M + h_k)$. In $(c, M + h_k)$, only the set $N_k(M)$ increases in one unit; thus, if $M_i \not\ni k$, the formula (1) for agent i is not affected. Then, the result follows. $\square$

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