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Relative bipolarization Lorenz curve

Gaston Yalonetzky
Leeds University Business School

Abstract

This note proposes a relative bipolarization quasi-ordering based on relative bipolarization (RB) Lorenz curves, which is consistent with the quasi-ordering generated by relative bipolarization indices satisfying key axioms. Therefore the quasi-ordering induced by RB Lorenz curves is identical to the one induced by relative bipolarization curves (the second-order curves in Foster and Wolfson (2010)). An appealing trait of RB Lorenz curves is their intuitive and straightforward representation of relative bipolarization situations, including minimum and maximum bipolarization, as well as any intermediate situation of perfect bimodality.

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Contact: Gaston Yalonetzky - g.yalonetzky@leeds.ac.uk.
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1 Introduction

The last couple of decades has witnessed substantial theoretical developments in the measurement of different polarization concepts. One such concept is that of bipolarization which was motivated, *inter alia*, by concerns over the disappearance of the middle class in developed countries. The seminal contribution for the measurement of bipolarization was an early version of Foster and Wolfson (2010), which introduced the first and second-order bipolarization curves. As Foster and Wolfson (2010) explain, bipolarization captures the degree to which societies may develop bimodal distributions above and below the median. Any progressive transfer that brings two people *on different sides of the median* closer together is deemed to reduce bipolarization; whereas any progressive transfer that brings two people *on the same side of the median* closer together is meant to increase bipolarization. Further significant contributions based on these two aspects of bipolarization were provided by Wang and Tsui (2000), Esteban, Gradin, and Ray (2007), Deutsch, Silber, and Hanoka (2007), Bossert and Schworm (2008), Chakravarty (2009), Lasso de la Vega, Urrutia, and Diez (2010), and Chakravarty and D'Ambrosio (2010).

Now, as Chakravarty (2009) explained, the plethora of bipolarization indices fulfilling desirable properties does not have to produce consistent rankings of distributions normally. Hence Chakravarty (2009) proved that bipolarization indices fulfilling the key axioms rank distributions consistently if and only if the relative bipolarization curves do not cross. That is, he provided the bipolarization-equivalent of the Lorenz consistency conditions.

This paper proposes an alternative relative bipolarization quasi-ordering, based on relative bipolarization (RB) Lorenz curves. For any given percentile, a traditional Lorenz curve is constructed by adding quantiles from the lowest to the highest, up to the given percentile's quantile, and dividing that total by the sum of all values (or, equivalently, dividing the incomplete mean up to that given percentile by the complete mean). By contrast, the RB Lorenz curve requires first splitting the ordered population into two halves separated by the median. Then a "half-percentile" set running from 0 to 0.5 is established (where 0 corresponds to the median itself). The next step is to relate every pair of quantiles equally ranked away from the median, i.e. one above and one below the median, to one single half-percentile. For instance the lowest value is paired with the highest value and they both constitute quantiles corresponding together to the 0.5 half-percentile. Finally, for any given half-percentile (running from 0 to 0.5), the RB Lorenz curve is constructed by adding the differences between each pair of quantiles, from the lowest to the highest difference, up to the given half-percentile, and dividing the total by the sum of all values (or, equivalently, dividing an incomplete mean quantile difference by the mean).

The note shows that the quasi-ordering produced by the RB Lorenz curves is identical to that of the relative bipolarization curves, and it also illustrates the appeal of the RB Lorenz curves in representing relative bipolarization situations including minimum and maximum bipolarization, as well as any intermediate situation of perfect bimodality. Moreover, unlike other bipolarization curves, *RB Lorenz curves do not involve the median* *for their derivation* (except, of course, to split the populations into two halves).

The rest of the note proceeds as follows. The next section provides the notation and a recapitulation of desirable properties for a bipolarization index. Special attention is given to normalization properties. Then the RB Lorenz curves are introduced and some bipolarization situations are illustrated with it. Thereafter the relative bipolarization quasi-ordering is shown to be necessary and sufficient for the consistency of relative bipolarization indices satisfying different sets of key desirable properties. The note ends with some concluding remarks, followed by the Appendix on BGG curves.

2 Preliminaries

2.1 Notation

Let y be a non-negative continuous variable from distribution Y, with quantile function: $y(p) : [0,1] \in \mathbb{R}_+ \to \mathbb{R}_+$. The population is divided into two halves: The bottom 50%, with $y \leqslant m$, where m is the median, and the top 50%, with $y \geqslant m$. Then quantile functions can be defined for each half:

$$y_L(p) \equiv y(0.5 - p) \; p \in [0, 0.5] \tag{1}$$

$$y_H(p) \equiv y(0.5 + p) \; p \in [0, 0.5] \tag{2}$$

For instance: $y_L(0) = y_H(0) = m$; $y_L(0.5) = \min y$; and $y_H(0.5) = \max y$. Finally, μ is the mean of the whole distribution.

For the description of the properties below, it also helps to define the following sets: N_U is the set of people above the median and N_L is the set of people below the median. The population has N individuals.

2.2 Desirable properties for a bipolarization index

The following definitions are helpful for the later introduction of desirable properties for a relative bipolarization index:

Definition 1. *Spread-decreasing Pigou-Dalton transfer (SDPD transfer): Distribution X is obtained from Y by an SDPD transfer if there is a pair of individuals (i,j), such that $i \in N_U \wedge j \in N_L$, $x_i = y_i - \delta$, $x_j = y_j + \delta$, for $\delta > 0$, $x_i \geqslant m \geqslant x_j$ and $x_k = y_k \forall k \neq (i,j)$.*

Definition 2. *Clustering-increasing Pigou-Dalton transfer (CIPD transfer): Distribution X is obtained from Y by an CIPD transfer if there is a pair of individuals (i,j), such that $i,j \in N_U \vee i,j \in N_L$, $y_i > y_j$, $x_i = y_i - \delta$, $x_j = y_j + \delta$, for $\delta > 0$, $x_i \geqslant x_j$ and $x_k = y_k \forall k \neq (i,j)$.*

Definition 3. *Minimum bipolarization (MIN): Distribution Y exhibits minimum bipolarization if and only if: $y_i = m \forall i \in N$.*

Definition 4. *Maximum bipolarization (MAX): Distribution Y exhibits maximum bipolarization if and only if: $y_i = 0 \forall i \in N_L \wedge y_j = 2m \forall j \in N_U$.*

While MIN is universally accepted as the benchmark of minimum bipolarization in the bipolarization literature , MAX does not carry consensus (e.g. note that the corresponding normalization axiom in Chakravarty (2009, p. 108) only relates to MIN). For instance, some of the bipolarization indices that can be derived from one of the families of Wang and Tsui (2000) do not exhibit a maximum at all. By contrast, the classic index by Foster and Wolfson (2010) reaches its maximum value of 1 if and only if MAX is present.[1] Having a maximum value that is well defined and easy to interpret can be advantageous in terms of comparability across distributions, but it comes at the cost of being unable to differentiate among all the different distributions characterized by the maximum definition (e.g. MAX in the case of bipolarization). Moreover, in the case of relative bipolarization measurement, MAX seems to be the only reasonable maximum benchmark since, once attained, bipolarization cannot be further increased either by progressive transfers on one side of the mean or by regressive transfers across the mean (e.g. reversing the SDPD). This situation occurs as long as everybody in the bottom half has zero income and everybody in the top half has the same positive income whichever its value. [2]

Now a bipolarization index, $\mathcal{I}(X) : \mathbb{R}_+^N \to \mathbb{R}_+$, is a continuous function that maps from an N-dimensional vector of real, non-negative numbers to the non-negative segment of the real line. Some bipolarization indices, e.g. FW, actually map onto the real interval $[0, 1]$. The desirable properties for a bipolarization index are listed below. The first two properties are widely used in the bipolarization literature.

Axiom 1. *SYM (Symmetry): If X is obtained from Y by multiplying the latter with a permutation matrix*[3] *then $\mathcal{I}(X) = \mathcal{I}(Y)$.*

Axiom 2. *POP (Population replication): If X is obtained from Y by replicating the latter's population by a constant factor of $\lambda \in \mathbb{N}_{++}$, then $\mathcal{I}(X) = \mathcal{I}(Y)$.*

Axiom 3. *SC (Scale invariance): If $X := (x_1, x_2, ..., x_N)$ and $Y := (kx_1, kx_2, ..., kx_N)$ where $k \in \mathbb{R}_{++}$, then $\mathcal{I}(X) = \mathcal{I}(Y)$.*

Relative bipolarization indices usually satisfy scale invariance (e.g. FW, the index of Deutsch et al. (2007), or the P_4^N of Wang and Tsui (2000)). However others have explored constructing bipolarization indices satisfying a less stringent property of unit consistency (e.g. Lasso de la Vega et al. (2010)) whereby only the ordinal comparisons, but not the indices' values, are unaffected by changes in the variable's unit of measurement. A third alternative is a property of translation invariance, which states that: if $X := (x_1, x_2, ..., x_N)$ and $Y := (x_1 + k, x_2 + k, ..., x_N + k)$ where $k \in \mathbb{R}$, then $\mathcal{I}(X) = \mathcal{I}(Y)$. The index P_3^N of Wang and Tsui (2000) is translation invariant.

[1]They propose the following index: $FW = (G_B - G_W)\frac{\mu}{m}$ where G_B and G_W are the between-group and within-group Gini indices, respectively. The two groups are top and bottom halves of the distribution.

[2]Interestingly, the alternatives of indices with, or without, a maximum value also exist in the inequality literature. For instance, the Gini coefficient takes its maximum value of $1 - 1/N$ if and only if $y = 0$ for everybody except for one single person for whom $y > 0$, *irrespective of how wealthy this person is*. By contrast, the mean log deviation, for instance, does not have a maximum value.

[3]The permutation matrix is a square N-dimensional matrix with entries of 0 and 1 such that all rows and columns add up to one.

The next two axioms restate the expected effects of SDPD and CIPD transfers on a bipolarization index:

Axiom 4. *SPREAD: If X is obtained from Y by an SDPD transfer, then $\mathcal{I}(X) < \mathcal{I}(Y)$.*

Axiom 5. *CLU: If X is obtained from Y by a CIPD transfer, then $\mathcal{I}(X) > \mathcal{I}(Y)$.*

Finally, two normalization axioms can be considered. Weak normalization (WNORM) states that the bipolarization index should attain its minimum if and only if X is characterized by MIN, whereas strong normalization (SNORM) states that the bipolarization index should attain its minimum and maximum values if and only if X is characterized by MIN and MAX, respectively:

Axiom 6. *WNORM (Weak Normalization): $\mathcal{I}(X) = 0$ if and only if X exhibits minimum bipolarization.*

Axiom 7. *SNORM (Strong Normalization): a) $\mathcal{I}(X) = 0$ if and only if X exhibits minimum bipolarization; and b) $\mathcal{I}(X) = 1$ if and only if X exhibits maximum bipolarization.*

3 The relative bipolarization (RB) Lorenz curve

The RB Lorenz curve is a function, $\psi(p)$ mapping from the continuous interval $[0, 0.5]$ (corresponding to the half-percentiles) to continuous interval $[0, 1]$. The construction of the curve requires considering the differences $y_H(p) - y_L(p)$ for every value of p in the $[0, 0.5]$ interval, i.e. the differences between pairs of y values, one above and one below the median, both equally ranked away *from the median*. Then the function is the following:

$$\psi(p) = \frac{\int_0^p [y_H(q) - y_L(q)]dq}{\mu} \; ; \; p \in [0, 0.5] \tag{3}$$

The RB Lorenz curve is characterized by the following properties, many of which are informative of particular features of bipolarization and inequality in the distribution:

Property 1. *Normalization: $0 \leqslant \psi(p) \leqslant 1$. In particular $\psi(0) = 0$.*

The maximum value of 1 is obtained for $p = 0.5$ if and only if MAX holds.

Property 2. *Minimum bipolarization: $\psi(p) = 0 \;\forall p$ if and only if $y = m \forall y$.*

That is, the curve is flat and overlaps with the horizontal axis if and only if all incomes are equal to the median.

Property 3. *Weak convexity: $\psi'(p) \geqslant 0$ and $\psi''(p) \geqslant 0$.*

The RB Lorenz curve is weakly convex just like a Lorenz curve. This is the case because the slope of the curve has the form: $\psi'(p) = \frac{y_H(p) - y_L(p)}{\mu}$, which is non-negative since $y_H(p) \geqslant y_L(p)$. Moreover, $\psi''(p) = \frac{1}{\mu}[\frac{\partial y_H(p)}{\partial p} - \frac{\partial y_L(p)}{\partial p}] \geqslant 0$ because $\frac{\partial y_H(p)}{\partial p} \geqslant 0$ and $\frac{\partial y_L(p)}{\partial p} \leqslant 0$.

Property 4. *Perfect bimodality: For any given pair of real non-negative numbers a and b such that $b > a$, $\psi(p) = 2\frac{b-a}{a+b}p$ if and only if $y_L(p) = a \forall p \in [0, 0.5]$ and $y_H(p) = b \forall p \in [0, 0.5]$.*

The RB Lorenz curve is a straight line if and only if the distribution is perfectly bimodal. With perfect bimodality: $\frac{\partial y_H(p)}{\partial p} = 0$ and $\frac{\partial y_L(p)}{\partial p} = 0$.

Property 5. *Maximum bipolarization:* $\psi(p) = 2p$ *if and only if* $y_L(p) = 0 \forall p \in [0, 0.5]$ *and* $y_H(p) = b \forall p \in [0, 0.5]$.

The RB Lorenz curve is a straight line with the highest possible slope if and only if MAX holds.

Property 6. *Between-group Gini: The between-group Gini, where the two non-overlapping groups are the bottom and the top halves divided by the median, is equal to* $\frac{1}{4}\psi(0.5)$. [4]

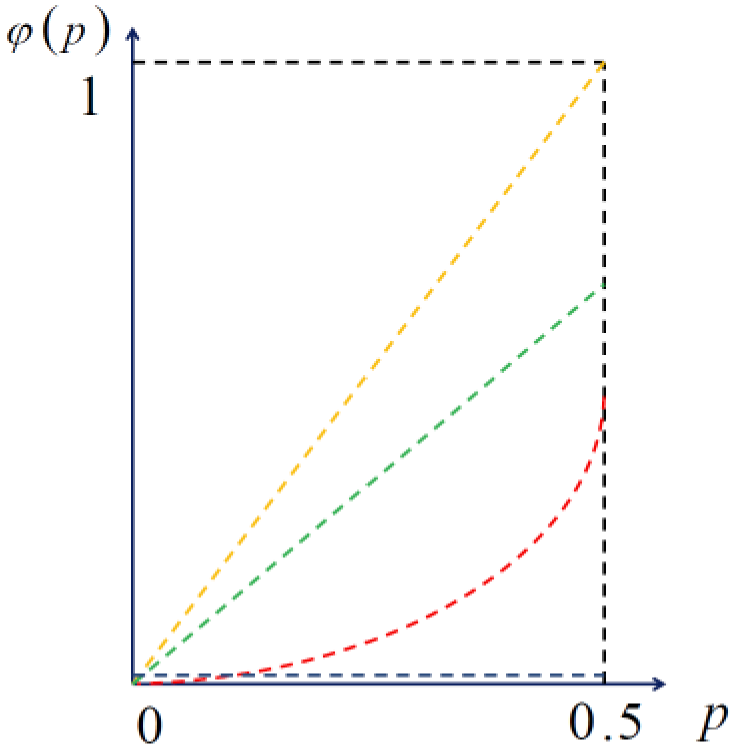
The intersection of the RB Lorenz curve with the right vertical axis (which intersects the horizontal axis at the value of 0.5) which is proportional to the between-group Gini index, provides a measure of the spread between the bottom and the top half of the distribution, in terms of the normalized difference of group means.

3.1 Graphical illustration

Figure 1 illustrates different situations of bipolarization using the RB Lorenz curves. MIN is represented by the dashed dark-blue line drawn just above the horizontal axis, but really overlapping with it. MIN also coincides with a situation of perfect equality. On the other extreme, MAX is represented by the dashed yellow-orange straight line running from the origin to the top right corner. Any other situation of perfect bimodality, aside from MAX, is represented by the dashed green straight line. Finally, a more common intermediate situation, away from both MIN and MAX, but also from perfect bimodality, is depicted by the dashed red convex curve.

[4] The definition of between-group Gini for two non-overlapping groups is based on Lambert and Aronson (1993).

Figure 1: Relative bipolarization quasi-orderings based on RB Lorenz curves: an illustration



3.2 Relative bipolarization quasi-ordering based on RB Lorenz curves

The following theorem states that all relative bipolarization indices satisfying, symmetry, population replication, scale invariance, spread-decreasing Pigou-Dalton transfers, and clustering-increasing Pigou-Dalton transfers, if and only if the RB Lorenz curves do not cross:

Theorem 1. *$\psi_A(p) \geqslant \psi_B(p)$ $\forall p$ and $\exists p|\psi_A(p) > \psi_B(p)$, if and only if distribution A exhibits more relative bipolarization than distribution B according to any relative bipolarization index satisfying SYM, POP, SC, SPREAD and CLU.*

Proof. [5] Showing that $\psi(p)$ fulfills SYM, POP and SC (and therefore is consistent with indices satisfying those properties) is straightforward. The purpose of the proof is to show, first, that $\psi_A(p) \geqslant \psi_B(p)$ $\forall p$ and $\exists p|\psi_A(p) > \psi_B(p)$ is necessary for $\mathcal{I}(A) > \mathcal{I}(B)$, for any $\mathcal{I}$ satisfying SPREAD and CLU. This is done by showing that if the RB Lorenz curves cross then one can always find a pair of relative bipolarization indices ranking A and B inconsistently, notwithstanding their fulfillment of SPREAD and CLU.

The next step is to show that the RB Lorenz curve condition is sufficient to imply the index-consistency condition: $\mathcal{I}(A) > \mathcal{I}(B)$, for any $\mathcal{I}$ satisfying SPREAD and CLU. The idea is to show that, given $\psi_A(p) \geqslant \psi_B(p)$ $\forall p$ and $\exists p|\psi_A(p) > \psi_B(p)$, we can always derive A from B (or viceversa), through an adequate sequence of CIPD and/or SDPD transfers. Since

[5] I would like to thank an anonymous referee for useful advice for the proof of Theorem 1.

the indices react to these transfers in pre-specified ways then the lack of RB Lorenz curve crossing suffices to guarantee ordering consistency across relative bipolarization indices satisfying SPREAD and CLU.

NECESSITY:

Imagine that the two RB Lorenz curves cross, that is: $\exists q | \psi_A(q) < \psi_B(q)$ and $\exists p | \psi_A(p) > \psi_B(p)$. Then, if we can find a pair of indices, $\mathcal{I}_1$ and $\mathcal{I}_2$, such that $\mathcal{I}_1(A) > \mathcal{I}_1(B)$ but $\mathcal{I}_2(A) < \mathcal{I}_2(B)$, we prove that the lack of curve crossing is necessary for different indices to rank consistently. Consider the following two distributions: $A := (10, 20, 30, 40, 50, 60, 70, 80, 90, 100)$ and $B := (10, 28, 30, 32, 50, 60, 70, 70, 90, 110)$. Their RB Lorenz curves cross. Now we consider two admissible indices from the family $P_2^N \equiv \frac{1}{m} \sum_{i=1}^{N} a_i y_i$ of Wang and Tsui (2000, p. 356). Our index $\mathcal{I}_1$ uses the following vector of a_i coefficients (for the incomes ranked in ascending order): $(-1, -5, -8, -10, -11, 11, 10, 8, 5, 1)$; and $\mathcal{I}_2$ uses the following coefficients: $(-1, -2, -4, -7, -11, 11, 7, 4, 2, 1)$. It is then straightforward to note that: $\mathcal{I}_1(A) > \mathcal{I}_1(B)$ but $\mathcal{I}_2(A) < \mathcal{I}_2(B)$. Hence, the fulfillment of the RB Lorenz curve condition is necessary for the fulfillment of the index-consistency condition.

SUFFICIENCY:

Let $\psi_A(p) \geqslant \psi_B(p)$ $\forall p$ and $\exists p | \psi_A(p) > \psi_B(p)$. We need to prove that, in such absence of curve-crossing, distribution A can be obtained from B through a series of CIPD and/or SDPD transfers.[6] Firstly, since A and B may have different means ($\mu(A)$ and $\mu(B)$, respectively), then we need to generate an auxiliary distribution, C, such that: $y_i^C = y_i^B \frac{\mu(A)}{\mu(B)}$. Since $\psi(p)$ fulfills SC, then $\psi_C(p) = \psi_B(p)$ $\forall p$, therefore: $\psi_A(p) \geqslant \psi_C(p)$ $\forall p$ and $\exists p | \psi_A(p) > \psi_C(p)$. Then the next step is to show that A can be obtained from C through the above transfers.

A necessary preliminary for the rest of the proof is Muirhead's theorem (Marshall, Olkin, and Arnold, 2011, p. 7-8). Let X and Z be the distributions of two non-negative, real-valued variables x and z, respectively. The population size of both distributions is n. The theorem, in its discrete form, says that: $\sum_{i=1}^{k} x_i \geqslant \sum_{i=1}^{k} z_i$ $\forall k \in [1, n] \subset \mathbb{N}_{++}$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} z_i$, with both x and y added from the lowest to the highest value, if and only if X can be obtained from Z through a sequence of rank-preserving progressive transfers.

Now, if $\psi_A(p) \geqslant \psi_C(p)$ $\forall p$ and $\exists p | \psi_A(p) > \psi_C(p)$, then either: $\psi_A(0.5) = \psi_C(0.5)$ or $\psi_A(0.5) > \psi_C(0.5)$. If $\psi_A(0.5) = \psi_C(0.5)$ then we can apply Muirhead's theorem, in its continuous-variable form, by defining quantile-gap variables for each distribution, e.g. $g^A(p) \equiv y_H^A(p) - y_L^A(p)$; and realizing that RB Lorenz curves accumulate these gaps ranked in ascending order. Then, given the relationship between $\psi_A(p)$ and $\psi_C(p)$, the theorem allows us to conclude that the distribution *of gaps* in A can be obtained from the distribution of gaps in C through sequences of rank-preserving progressive transfers. This means transfers in which some $\delta > 0$ amount is transferred from $g(r)$ to $g(q)$, such that $q < r$. It turns out that only two types of transfers between incomes (or a combination thereof) would produce the require transfers between gaps: i) A CIPD transfer from $y_H(r)$ to $y_H(q)$; ii) A CIPD transfer from $y_L(q)$ to $y_L(r)$. Therefore, since A can be obtained from C through a sequence of CIPD transfers, and C is obtained through a multiplicative rescaling of B, it

[6] Reverse SDPD transfers in our example, i.e. spread-increasing, regressive transfers across the median.

must be the case that: $\mathcal{I}(A) > \mathcal{I}(C) = \mathcal{I}(B)$, for all relative bipolarization indices satisfying SC and CLU.

But what if $\psi_A(0.5) > \psi_C(0.5)$? In this case Muirhead's theorem does not apply directly. However if we can derive a distribution D from C, such that: $\psi_A(p) \geqslant \psi_D(p)\ \forall p$, $\exists p|\psi_A(p) > \psi_D(p)$ and $\psi_A(0.5) = \psi_D(0.5)$, then we can apply the theorem to the comparison between A and D. It turns out that we can derive D from C through a sequence of reverse SDPD transfers, i.e. *spread-increasing*, Pigou-Dalton transfers across the median. The natural starting point is to set $g^D(p) = g^C(p) \quad \forall p < 0.5$ and render $g^D(0.5) = g^C(0.5) + \delta$ (with $\delta > 0$) through a reverse SDPD transfer from $y_L^C(0.5)$ to $y_H^C(0.5)$ until $\psi_A(0.5) = \psi_D(0.5)$. If δ is not enough (e.g. because $y_L^D(0.5)$ is now 0 and it is still not the case that $\psi_A(0.5) = \psi_D(0.5)$), then an additional reverse SDPD transfer is implemented with the gap of C immediately below $p = 0.5$, and so on, until $\psi_A(0.5) = \psi_D(0.5)$. This sequence of reverse SDPD transfers ensures both that $\psi_A(0.5) = \psi_D(0.5)$ and that ψ_D is never above ψ_A. Then we apply Muirhead's theorem and conclude that A can be obtained from D through a sequence of CIPD transfers. To summarize, C is obtained from B through multiplicative rescaling, D is obtained from C through spread-increasing, regressive transfers (i.e. reverse SDPD transfers across the median), and A is obtained from D through CIPD transfers (each on one side of the median).

Hence, whenever $\psi_A(p) \geqslant \psi_B(p)\ \forall p$ and $\exists p|\psi_A(p) > \psi_B(p)$, we can always obtain A from B through a sequence of CIPD transfers, reverse SDPD transfers, and/or multiplicative rescaling transformations. Since relative bipolarization indices increase with these transfers (and fulfill SC) it must be the case that: $\psi_A(p) \geqslant \psi_B(p)\ \forall p$ and $\exists p|\psi_A(p) > \psi_B(p)$ suffices to imply that $\mathcal{I}(A) > \mathcal{I}(B)$ for all indices satisfying SPREAD, CLU and SC.

In conclusion, whenever it is the case that $\psi_A(p) \geqslant \psi_B(p)\ \forall p$ and $\exists p|\psi_A(p) > \psi_B(p)$, we can always obtain A from B through a sequence of CIPD and/or reverse SDPD transfers, all of which lead to $\mathcal{I}(A) > \mathcal{I}(B)$, for any $\mathcal{I}$ satisfying SPREAD and CLU. ■

Applying theorem 1 to the hypothetical distributions depicted in figure 1, it is clear that any pairwise comparison among them is robust since the curves do not cross. Moreover, using the colours to identify the distributions, the quasi-ordering in terms of increasing bipolarization becomes: dark-blue, red, green, and yellow-orange. All relative bipolarization indices satisfying the key properties mentioned above rank those four distributions consistently.

4 Conclusion

This paper introduced a relative bipolarization quasi-ordering based on relative bipolarization (RB) Lorenz curves. The curves provide an intuitive illustration of different bipolarization situations within the confines of a 0.5x1 rectangle in quadrant I of the Cartesian coordinate system. While the intersection of the RB Lorenz curve with the vertical right axis provides a measure of the maximum relative spread between the means of the top and bottom half of the distribution (i.e. when everybody is accounted for), the non-negative slope of the curve depends on the degree of clustering within both halves. The higher the clustering the more linear the slope becomes.

As shown above, indices satisfying the key axioms of relative bipolarization rank distributions consistently if and only if the RB Lorenz curves do not cross. Therefore the RB Lorenz curve quasi-ordering is actually identical to the quasi-ordering of relative bipolarization curves characterized by Chakravarty (2009).

5 Acknowledgments

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References

Bossert, W. and W. Schworm (2008). A class of two-group polarization measures. *Journal of public economic theory 10*(6), 1169–87.

Chakravarty, S. (2009). *Inequality, Polarization and Poverty. Advances in distributional analysis*. Springer.

Chakravarty, S. and C. D'Ambrosio (2010). Polarization orderings of income distributions. *Review of Income and Wealth 56*(1), 47–64.

Deutsch, J., J. Silber, and M. Hanoka (2007). On the link between the concepts of kurtosis and bipolarization. *Economics Bulletin 4*(36), 1–5.

Esteban, J., C. Gradin, and D. Ray (2007). An extension of a measure of polarization, with an application to the income distribution of five OECD countries. *Journal of Economic Inequality 5*(1), 1–19.

Foster, J. and M. Wolfson (2010). Polarization and the decline of the middle class: Canada and the US. *Journal of Economic Inequality 8*(2), 247–73.

Lambert, P. and R. Aronson (1993). Inequality decomposition analysis and the Gini coefficient revisited. *The Economic Journal 103*(420), 1221–7.

Lasso de la Vega, C., A. Urrutia, and H. Diez (2010). Unit consistency and bipolarization of income distributions. *Review of Income and Wealth 56*(1), 65–83.

Marshall, A., I. Olkin, and B. Arnold (2011). *Inequalities: Theory of Majorization and its applications* (Second ed.). Springer Series in Statistics. Springer.

Wang, Y.-Q. and K.-Y. Tsui (2000). Polarization orderings and new classes of polarization indices. *Journal of Public Economic Theory 2*(3), 349–63.