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Some λ-Separable Frisch Demands with Utility Functions

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We complete the characterization of two Frisch demand systems first developed by Browning, Deaton and Irish (1985), and show that that these systems (i) do not restrict intertemporal substitution; but (ii) imply momentary utility functions which are additively separable in consumption. These utility functions turn out to take the well-known exponential and Stone-Geary forms.

1. Introduction

Frisch demands are expressed as functions of prices and a positive quantity λ, related to the consumer's marginal utility of expenditures. In a much cited paper which first established many of the general properties of the Frischian system, Browning, Deaton, and Irish (1985) (henceforth, BDI) describe two particular Frisch demand systems which are additively separable in λ. Such λ-separable demand systems are very useful in empirical applications because they permit one to control for consumer wealth or permanent income using a simple linear latent-variable approach (e.g., Blundell, Browning, and Meghir 1994). In addition to BDI, Browning (1986) and Blundell, Fry, and Meghir (1990) investigate some of the properties of these two systems, but their characterization is incomplete.

In this paper we provide a complete characterization of the two demand systems discussed in BDI. Though BDI's contribution is now three decades old, there is a resurgent interest in the use of Frisch demand systems (and especially λ-separable demands) in applications in several fields of economics. For example, there is important recent work on non-homothetic Frisch demand systems in trade (e.g., Mrázová and Neary 2014), and a large body of recent work in labor economics has occupied itself with understanding the Frisch elasticities estimated by BDI (Keane and Rogerson 2012, offer a survey and discussion).

The defining characteristic of BDI's two demand systems is that the first ("Case 1") has *quantities* additively λ-separable; the second ("Case 2") has *expenditures* additively λ-separable. Browning (1986) calculates the expenditure functions implied by these two demand systems under a particular cardinalization of the momentary utility function, and compares two different features: (i) intra-temporal substitution possibilities (i.e., among goods consumed in a particular period); and inter-temporal substitution elasticities.

Browning (1986) and Blundell, Fry, and Meghir (1990) observe that Case 1 demands have quite restrictive intra-temporal substitution possibilities, and "unattractive" intertemporal substitution elasticities. Case 2 demands fare better in this account on both intra- and inter-temporal dimensions. One contribution of the present paper is that we're able to show that though intra-temporal substitution possibilities are indeed very limited, that these particular Frisch demands impose *no* restrictions on intertemporal substitution. To suppose otherwise is to forget that Frisch demand systems are by their nature not invariant to monotonic transformations of utility, a mistake Frisch (1959) characterizes as a "heresy".

Though others have observed (MaCurdy 1983; Blundell, Fry, and Meghir 1990; Blundell, Browning, and Meghir 1994) that the intertemporal substitution elasticity depends on the cardinalization of momentary utility which one adopts, here we develop this result, demonstrating that in fact neither of these λ-separable Frisch demands restricts the intertemporal substitution elasticity at all. However, we also show that both of the λ-separable demand systems considered by BDI imply additively separable utility; thus both intra- and inter-temporal substitution possibilities will depend on some monotonic transformation of an additively separable momentary utility function.

2. Characterization of the General Consumer's Problem

We begin by developing some notation and results which do *not* rely on λ-separability.

Consider the consumer in the standard life-cycle model. In addition to standard restrictions on the utility function (increasing, concave, continuously twice differentiable), the utility of this consumer is additively separable across both time and states. This allows one

to specify the demand behavior the consumer as a two-stage budgeting problem (Gorman 1959): in the first stage, the consumer budgets total expenditures at each date and state; then given this budget the consumer chooses among n different consumption goods which affect utility just at that date-state.

2.1. **Second Stage Budgeting.** Consider this second-stage problem of allocating resources within a particular date-state with a budget x facing prices $p \in \mathbb{R}_+^n$. A consumer with a momentary utility function $U : \mathbb{R}^n \to \mathbb{R}$ (assumed strictly increasing and strictly concave) over a vector $c \in \mathbb{R}^n$ of goods and services will solve

$$V(p, x) = \max_c U(c) \qquad \text{such that } p^\top c \leq x; \tag{1}$$

this defines the usual Marshallian indirect utility function V; the first order conditions define the Frisch demands. To see this, let $u_i = \partial U / \partial c_i$ be the marginal utility of consumption of good i, and let $u = (u_1, \ldots, u_n)$ be the vector of these marginal utilities. Then the solution to (1) is characterized by a set of n first order conditions

$$u(c) = \lambda p, \tag{2}$$

where λ is the Karush-Kuhn-Tucker multiplier associated with the budget constraint in (1). Inverting (2) yields the Frisch representation of demands

$$c = f(\lambda, p). \tag{3}$$

The form of the Frisch demands f is determined by the form of the marginal utilities u.

BDI rely on a dual representation of the consumer's problem that involves treating the consumer as though a profit-maximizing firm, with the utility function U playing the role of the firm's production function. This then defines a profit function $\pi(1/\lambda, p) = U(f(\lambda, p))/\lambda - p^\top f(\lambda, p)$. We rely on the following useful facts about the profit function π and Frisch demand systems (BDI report these and others):

Remark 1. *Browning, Deaton, and Irish (1985) collect the following properties of Frisch demands:*

(1) *π is twice continuously differentiable.*
(2) *$-\partial \pi / \partial p_i = c_i = f_i(\lambda, p)$.*
(3) *$-\partial \pi / \partial(1/\lambda) = U(f(\lambda, p))$.*
(4) *$f(\lambda, p)$ is homogeneous of degree zero in $(1/\lambda, p)$.*

2.2. **First Stage Budgeting with V.** Now we turn our attention to the first stage of the consumer's budgeting problem. If the intertemporal utility function for the consumer takes the form

$$\mathrm{E}_0 \sum_{t=1}^T \beta^t V(x_t, p_t), \tag{4}$$

(where the V that appears in (4) is the indirect utility function that appears in (1), and where E_0 denotes the time zero expecations operator), then the marginal utility of expenditures at t is $\lambda_t \equiv \partial V(x_t, p_t)/\partial x$. The elasticity of this marginal utility with respect to x is equal to minus the consumer's relative risk aversion (RRA) (Pratt 1964), which in turn is equal to the reciprocal of the intertemporal substitution elasticity (ϕ), or

$$\frac{\partial^2 V/\partial x^2}{\partial V/\partial x} x = -\mathrm{RRA} = 1/\phi;$$

alternatively, $1/\phi$ is identical to what Frisch (1959) calls "money flexibility".

2.3. Frisch Demands are not Invariant to Monotonic Transformations.

Here we must state the obvious: Marshallian demands must be invariant to any monotonic transformation of momentary utility, and any observable behavior on the part of the consumer can only identify the momentary utility function in (1) up to a monotonic transformation M. This extends to the indirect utility function, which can be known only up to $M(V(p,x))$, and to the multiplier on the budget constraint, which can be known only up to $\lambda \partial M(U)/\partial U$ (this last provided that M is differentiable).

This invariance property of Marshallian demands does not extend to Frisch demands. From (1), if a consumer with momentary utility U in has Frisch demands $f(\lambda, p)$, then an otherwise identical consumer with momentary utility $M(U)$ will have Frisch demands

$$f^M(\lambda, p) \equiv f\left(\lambda \frac{\partial M}{\partial U}(U(f(\lambda, p))), p\right). \tag{5}$$

A salient consequence is that if the momentary utility function U has Frisch demands f which are separable in λ, the *same demand behavior* will be consistent with a utility function $M(U)$ and Frisch demands f^M. However, f^M will generally *not* be separable in λ.

3. Some λ-separable demands

We now consider the separability requirements of BDI. In the first case *quantities* demanded are additively separable in λ; or

$$c_i = f_i(\lambda, p) = a_i(p) + b_i(\lambda) \tag{6}$$

for some (non-constant) functions a_i and b_i. In the second case it's *expenditures* that are λ-separable, or

$$pf_i^*(\lambda, p) = a_i^*(p) + b_i^*(\lambda), \tag{7}$$

where again a_i^* and b_i^* are assumed to be non-constant.

3.1. Quantities λ-separable.

BDI showed that the profit functions for the first case, with demands satisfying (6), can be written

$$\pi(\lambda, p) = \alpha/\lambda + D(p) + \sum_{i=1}^{n} \beta_i p_i \log(p_i \lambda) \tag{8}$$

where $D(p)$ is some linearly homogeneous function of prices. Using the second property of Remark 1, from this profit function BDI obtain the demands

$$f_i(\lambda, p) = -d_i(p) - \beta_i(\log(p_i \lambda) + 1), \tag{9}$$

where $d_i(p) = \partial D/\partial p_i$. At the same time, applying property 3 of Remark 1 implies that utility does not depend on $d_i(p)$. It follows that the utility function U and Frisch demands f satisfy

$$U(f(\lambda, p) + d(p)) = \alpha - \lambda \sum_{i=1}^{n} \beta_i p_i;$$

3

this cancels out the vector $d(p)$ from demands (9), leaving utility independent of $d(p)$ as property 3 requires. It's then easy to verify that the demands (9) are the solution to the consumer's Lagrangian problem

$$\max_c U(c) + \lambda(D(p) - p^\top c)$$

when the consumer's utility function takes the exponential form

$$U(c) = \sum_{i=1}^{n} -\beta_i \exp\left(-\frac{c_i}{\beta_i}\right). \tag{10}$$

Note that the utility function is additively separable across all goods, and thus features no specific substitution possibilities; the term $D(p)$ which appears in this problem has a natural interpretation as the consumer's *income*, or price-dependent endowment. Though in the classical formulation the consumer's problem income is a single quantity, that problem generalizes naturally to the case in which income depends on prevailing prices.

3.2. Expenditures λ-separable.

BDI also show that the profit function for demands satisfying (7) can be written

$$\pi^*(\lambda, p) = \alpha^*/\lambda + D^*(p) - \frac{1}{\lambda}\sum_{i=1}^{n}\beta_i^* \log(p_i\lambda), \tag{11}$$

with $D^*(p)$ linear homogeneous.

Using a strategy similar to that we've used in the first case, this implies Frisch demands

$$f_i^*(\lambda, p) + d_i^*(p) = \beta_i^*/(\lambda p_i). \tag{12}$$

Again, the vector $D^*(p)$ can be regarded as a price-dependent endowment, and again application of the envelope theorem to (11) implies that utility must be written as $U(f^*(\lambda, p) + d^*(p))$.

An argument identical in form to that we've used above leads to the conclusion that the demands in this case are generated by a utility function that takes the Stone-Geary form:

$$U^*(c) = \sum_{i=1}^{n}\beta_i^* \log(c - d_i(p)), \tag{13}$$

save that where the usual Stone-Geary formulation takes $d(p)$ to be a vector of constants, this formulation allows it to be a (homogeneous degree zero) function of prices. This utility function was used by Stone and Geary to construct the (Marshallian) linear expenditure system; it's interesting that it also generates linear expenditures in the Frischian representation.

4. λ-Separable Demands and the Intertemporal Elasticity of Substitution

Browning, Deaton, and Irish (1985) show that having intra-temporal Frisch demands separable in λ implies a particular utility function U and indirect utility function V, but this implies only that the consumer's intertemporal utility takes the form

$$\mathrm{E}_0 \sum_{t=1}^{T} \beta^t M(V(x_t, p_t)), \tag{14}$$

not (4), where we should now think of M as a 'cardinalization' of utility which can't be identified from purely intratemporal demand behavior. M is necessarily increasing; assume

for simplicity that M is also twice continuously differentiable. Note however that M need not be concave.

The marginal utility of expenditures at t for a consumer with cardinal momentary utility $M(V)$ is $\lambda_t^M \equiv \partial M(V(x_t, p_t))/\partial x = M'(V)V'(x_t, p_t)$. The elasticity of this marginal utility with respect to x is, as before, the reciprocal of the consumer's intertemporal elasticity of substitution, or

$$1/\phi^M = x \left(\frac{\partial^2 M/\partial U^2}{\partial M/\partial U} + \frac{\partial^2 V/\partial x^2}{\partial V/\partial x} \right). \tag{15}$$

Browning (1986) and Blundell, Fry, and Meghir (1990) argue that when quantities are λ-separable as in (6), then the consumer's ϕ at t is given by

$$\phi(x_t, p_t) = x_t/\beta(p_t), \tag{16}$$

and thus that the intertemporal elasticity of substitution is increasing in expenditures, which they deem an unattractive property. This argument implicitly assumes the particular cardinalization of momentary utility (10).

However, we've already seen that ϕ is not invariant to monotonic transformations of the momentary utility function. If Frisch demands satisfy (6), then from (16) we have

$$1/\phi^M = x \left(\frac{\partial^2 M/\partial U^2}{+} \frac{1}{\beta(p)} \right).$$

Accordingly, though λ-separability restricts the curvature of the utility function U, it does *not* restrict the intertemporal elasticity of substitution or the relative risk aversion of a consumer with preferences given by (14).

Of course, some care must be taken if one is to choose a particular parametric form for the monotonic transformation M. For example, choosing $M(U) = \frac{U^{1+\sigma}}{1+\sigma}$ would imply

$$1/\phi^M = \left(\sigma + \frac{x}{\beta(p)} \right),$$

which does nothing to address the concern that ϕ is decreasing in total expenditures.

5. Conclusion

We've shown that assuming that intratemporal Frisch demands are λ-separable places no restrictions on the consumers' intertemporal elasticity of substitution, answering one of the complaints of Browning (1986) and Blundell, Fry, and Meghir (1990). It's the assertion of a particular cardinalization of momentary utility that determines the intertemporal trade-offs that consumers face, not the form of the momentary utility function.

We've also shown that the two particular λ-separable demands investigated by Browning, Deaton, and Irish (1985) can be generated by utility functions that are additively separable—there are no specific substitution possibilities in these demand systems. Further, since these two demand systems are also quasi-homothetic, they also place extreme restrictions on Engel curves, which must be linear.

These two λ-separable demand systems place stronger restrictions on intratemporal demands than is generally called for, and though they put no more restrictions on intertemporal substitution possibilities than any other intertemporally separable demand system, neither do they put fewer. We conclude that these two demand systems are too restrictive for most serious empirical applications.

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