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Efficient Sets Are Very Small

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Abstract

We show that Efficient sets are not just small but very small. In fact we show that efficient sets have measure zero with respect to every continuous sigma-finite product measure on R^p.

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1 Introduction

It has been over a century since Vilfredo Pareto [3] first introduced the concept of efficiency and it continues to play a significant role in problems of economic analyses and constrained maximization. The notion of efficiency is an integral part of the process of choices made by consumers and producers in all types of markets and situations. Efficient sets are relevant in level sets of utility functions, quasi-indifference classes associated with a preference relation, mean-variance frontiers, production possibility frontiers, data envelopment and portfolio selection. Beardon and Rowat in [1] introduced efficient sets and showed several interesting results. In an effort to find out how small efficient sets are, using the Lebesgue density theorem, Beardon and Rowat showed that every efficient set in R^p is of Lebesgue measure zero. They also showed that the Hausdorff dimension of any efficient set in R^p is no more than $p-1$. In this paper we strengthen the first result by showing that every efficient set has measure zero for any continuous σ-finite product measure on R^p. This shows that efficient sets are very small. By a simple example, we shall also show that they need not be universally null if $p > 1$. We conclude the paper with some remarks and a geometric characterization of efficient sets in R^2.

Definition [1]: A subset A of R^p is called an efficient set if A does not have two points $\underline{x} = (x_1, x_2, \cdots, x_p)$ and $\underline{y} = (y_1, y_2, \cdots, y_p)$ such that $x_i < y_i$ for all i. We shall write $x << y$ if $x_i < y_i$ for all i.

The empty set by definition efficient. All singleton sets in R^1 are efficient and these are the only efficient sets in R^1. Any vertical line or horizontal line or any line with a negative slope (or subsets there of) in R^2 is efficient.

We shall observe a special property of sections of efficient sets. We are not going to say that sections of efficient sets are efficient, which would have been nice. Consider a vertical line at 0 (on the x-axis) in R^2. This set is efficient, though the section at 0 (on the x-axis) is the whole of R^1 and is not efficient in R^1.

This being said, we shall observe a property of the family of all x- sections of efficient sets in Corollary 2, which itself follows from Theorem 1.

Theorem 1: A is an efficient set of $R^p(= R^1 \times R^{p-1}$, say) if and only if the sections $\{A_x; x \in R^1\}$ have the property that whenever $x_1 < x_2$ are two points in R^1, no point in A_{x_1} is $<<$ any point in A_{x_2}. Here $A_x = \{\underline{y} : (x, \underline{y}) \in A\}$, the x-section of A.

Proof: Suppose that $x_1 < x_2$ and that A is efficient. If $\underline{y_1}$ is a point inA_{x_1}, $\underline{y_2}$ is a point in A_{x_2}, and $\underline{y_1} << \underline{y_2}$, then clearly the point $(x_1, \underline{y_1}) << (x_2, \underline{y_2})$ and both points are in

A. This contradicts the efficiency of A. The converse is clear.

Corollary 2: If A is an efficient set of $R^p(= R^1 \times R^{p-1}$, say), then, whenever $x_1 < x_2$ are two points in R^1, $A_{x_1} \bigcap A_{x_2}$ is efficient.

Proof: If $A_{x_1} \bigcap A_{x_2}$ is not efficient, there would exist points $\underline{y_1} << \underline{y_2}$ in $A_{x_1} \bigcap A_{x_2}$. This is not possible by Theorem 1.

We remark that the converse of the above corollary is not true. $A = (\{0\} \times [2,3]) \cup (\{1\} \times [1,2]) \cup (\{0\} \times [3,4])$ satisfies the hypothesis of corollary 2 but is not efficient.

By a measure we mean a nonnegative possibly infinite countably additive set function defined on the Borel $\sigma-$ field of R^p. We shall call a measure μ, a continuous measure, if $\mu(\{x\}) = 0$ for every singleton x in R^p. We shall call a measure μ, a $\sigma-$ finite measure, if there exist countably many Borel sets $B_i : i \geq 1$ such that $\bigcup B_i = R^p$ and $\mu(B_i) = 0$. Several results in measure theory are true for σ-finite measures. For example, Fubini theorem and Tonelli's theorem are true for such measures. Another property satisfied by all σ-finite measures is the property of countable chain condition, i.e., if μ is a σ-finite measure, any family of Borel sets $\{B_i, i \in I\}$ such that for any $i \neq j$ in I, $\mu(B_i \cap B_j) = 0$ has the property that $\{i : \mu(B_i) > 0\}$ is a countable family. See [4] or [2].

Let us use the above corollary to show that any efficient set in R^p is a set of measure zero for any product of σ-finite continuous measures, and not just for the Lebesgue measure. If $\mu_1, \mu_2, \cdots, \mu_p$ are σ-finite measures on $R_1, R_2, \cdots, R_p$, all being copies of the real line R, then we shall write μ for the product measure $\mu_1 \times \mu_2 \times \cdots \times \mu_p$. Clearly μ is σ-finite. If each μ_i is continuous in the sense that $\mu_i(\{x\}) = 0$ for every x in R_i, then μ is also continuous. We shall call any such μ, a σ-finite continuous product measure. The Lebesgue measure on R^p is an example of such a measure.

Theorem 3: If A is an efficient set in R^p, then for any σ-finite continuous product measure μ on the Borel σ-field of R^p, $\mu(A) = 0$. Thus any efficient set is very small.

Proof : We shall prove this theorem by induction on p. Clearly the result is true for $p = 1$. We assume without loss of generality that A is closed, since, as was shown in [1] if A is efficient, its closure $\overline{A}$ is also efficient.

Let $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_p$ be a σ-finite continuous product measure on the Borel σ-field of $R^p = R^1 \times R^{p-1}$. Let η on the Borel σ-field of R^{p-1} be defined by $\eta(B) = \mu_2 \times \cdots \times \mu_p(B)$. Observe that for any Borel set A in R^p, $\mu(A) = \int \eta(A_x)\, d\mu_1(x)$ by Fubini's and Tonelli's theorems.

Now we shall show that there are at most countably many $x's$ such that $\eta(A_x)$ is nonzero. This will imply, since μ_1 is continuous, that $\mu(A) = 0$.

Take any two distinct points x_1 and x_2 in R^1. By corollary 2, $A_{x_1} \cap A_{x_2}$ is efficient. By the induction hypothesis, $\eta(A_{x_1} \cap A_{x_2}) = 0$. Thus $\{A_x : x \in R^1\}$ satisy the condition that the intersection of any pair of these sets is of η measure zero. Hence, by the countable chain condition, at most countably many sets in $\{A_x : x \in R^1\}$ are of positive η measure. But for these countably many x's $\mu_1(\{x\}) = 0$. For the rest of the x's $\eta(A_x) = 0$. By the integral identity above, $\mu(A) = 0$.

Remark 4:. We have shown that every efficient set in R^p is of measure zero for every product of nonnegative continuous σ- finite measures. However, it need not be of measure zero for every continuous measure in R^p when $p > 1$. Note that a subset B of R^p is called a universally null set if $\tau(B) = 0$ for every nonnegative continuous σ- finite measure τ. We are saying that for $p > 1$, efficient sets need not be universally null. For example, any vertical line, horizontal line, or a line with negative slope in R^2 is not universally null. However, in R^1 every efficient set (singleton set) is universally null.

Now, there are other classes of nonatomic measures on R^p. For example, one could consider the family of measures on R^p which give positive measure to all open sets. One could also consider the family of measures on R^p that give measure zero to every hyperplane. From the definition of an efficient set on would suspect that every efficient set is of measure zero for every measure in these classes. Clearly the answer to these questions is positive if $p = 1$. In the following two remarks we shall show that even for $p = 2$, the answer is in the negative.

Remark 5: There is an efficient set E in R^2 and a measure μ on R^2 which gives positive measure to every nonempty open set such that $\mu(E) \neq 0$. That is to say that efficient sets need not be of measure zero for all measures that give positive measure to all noempty open sets. This can be seen as follows. Let $\tau = \lambda + \mu$ where λ is the Lebesgue measure on R^2 and μ is the measure on R^2 concentrated on the vertical line at, say, 1 on the x-axis, similar to Lebesgue measure on R.Then τ is a measure on R^2 which gives positive measure to every open set in R^2. This is because the Lebesgue measure λ has this property. If we consider the efficient set $E =$ the vertical line at 1 on the $X-$ axis, then $\tau(E) \neq 0$.

Remark 6: There is an efficient set E in R^2 and a measure μ on R^2 which gives measure zero to every hyperplane such that $\mu(E) \neq 0$. That is to say that efficient sets need not be of measure zero for all measures that give measure zero to all hyperplanes. This can be seen as follows. Take a circle of radius 1 at $(2, 2)$ in R^2. Consider the linear measureη concentrated on the circumference of the circle. Then $\eta(A) = 0$ for every hyper plane A in R^2. If we look at the set E of points on a quarter of the circle facing $(0, 0)$, then this set is efficient and is of positive measure with respect to η.

Now we shall give a complete geometric characterization of the efficient sets in R^2. From the proof of theorem 1, if E is an efficient set in R^2 and x_1 and x_2 are points on the x-axis, it is clear that $E_{x_1} \cap E_{x_2}$ is either the empty set or a singleton set.

Lemma 7: Let E be an efficient set in R^2. For any z on the X-axs if E_z is nonempty let $a_z = inf$ points in E_z and $b_z = sup$ points in E_z. If x_1 and x_2 are two points on the X-axis with $x_1 < x_2$, then, $a_{x_1} \geq b_{x_2}$.

Considering the version of lemma 7 for the y-sections of E we can give a structure theorem for efficient sets in R^2. For sets C and D in R we shall write $C < (\leq) D$ to mean that every point of C is $< (\leq)$ every point of D. Here is the geometric characterization of efficient sets in R^2.

Theorem 8: Let E be an efficient set in R^2. Then there is an index set I and pairwise disjoint families $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ such that for every i either A_i is a singleton set or B_i is a singleton set or both and if $i \neq j$, either $A_i < A_j$ and $B_i \geq B_j$, or $A_i > A_j$ and $B_i \leq B_j$ and $E = \cup(A_i \times B_i)$.

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