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Chaos in Lebanese GDP: The Lorenz Attractor Approach

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Abstract

In this paper we will refer to the chaos theory to analyze the evolution of Lebanese GDP growth rate. We use macroeconomic series as GDP, household consumption expenditure and investment over the period of 1970-2017 to estimate the coefficients of the differential equations of the Lorenz system. We found that the evolution of the Lebanese GDP growth looks like a strange attractor generated by chaotic dynamic.

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1. Introduction

Lebanon is a country that has experienced several civil conflicts that have had a significant impact on the evolution of the Gross Domestic Product (GDP) especially during the period of 1975-1990. Therefore, from 1970 to 2017, Lebanon has known stabilities periods (peace periods) and conflict periods. These unstable political situations can explain why the evolution of GDP growth rate seems chaotic.

GDP fluctuations are marked substantially during the war period but become weaker after 1990. Before the civil war, the GDP showed a growth rate of 11% in 1972. Then, this rate fell drastically to – 83% in 1976 and increased to 60% the next year (United Nations, 2019). This instability continued until the end of the civil war in 1990. In fact, the economic fluctuations are showing different magnitude during and after the civil war and the Lebanese economic system seems to behave in a chaotic manner. Indeed, the others macroeconomic variables such as household consumption expenditure and gross fixed capital formation (investment) for instance follow the same movement during the period 1970-2017.

Chaos theory is primarily used in the meteorology fields (Lorenz, 1960, 1972). The main insight behind this concept is that even simple deterministic system can sometimes produce unpredictable situations notably when such deterministic system has a sensitivity to initials conditions in the short run. This theory involves the strange attractor concept to which the trajectories of a variable have a bizarre structure being nether simple smooth, nor continuous curves but fractals (Puu, 1997). Fractals (Mandelbrot, 1982) could be an indefinite set of unconnected points or a smooth curve with mathematical discontinuity or curve that is fully connected but discontinuous everywhere.

The inclusion of the chaos theory in economic analysis is not recent. For example, Mandelbrot (1963) analyses the chaotic variation of speculative prices. Kesley (1988), using the overlapping generation model, assert that economics models involve chaos. Baumol and Benhabib (1989) present what is the chaos, how it works and whether it does or does not occur in economic phenomena. Viad and al., (2010), taking an example of chaos in exchange rates, show that chaos theory is related with the notion of nonlinearity. Thus, the chaos theory presents an advantage with respect to time series methods because the main macroeconomic aggregates (which reflect economic behaviors) are mostly governed by non-linear dynamic.

In our paper, we propose an empirical application of the Lorenz equations system to looking for a chaotic behavior in the Lebanese GDP by types of expenditure (at purchasers' price including notably consumption expenditures and investment).

To analyze these chaotic characteristics of the Lebanese economy, the present paper contains the following sections. Section 2 presents the characteristic of the Lebanese GDP in the phase space. Section 3 proposes an econometric estimation of the Lorenz system by taking the Lebanese macroeconomic data over the period of 1970-2017 and shows a simulation of the Lorenz attractor. Section 4 concludes and comments.

2. Characteristics of the Lebanese GDP in the space phase

We use the Brock, Dechert, Scheinkman and Lebaron (BDS) test (1996) to analyze the nonlinearity of the Lebanese GDP. This series has been unloaded from United Nations Statistic (2019) (at constant US dollars price). We write it in logarithm and in first difference (and obtain the GDP growth rate). Such test is a tool for detecting the null hypothesis of independence and identically

distributed (*i.i.d*) against an unspecified alternative. This test is a necessary but not sufficient condition to confirm the chaotic behavior of the time series. The BDS test indicates if the series is linear or not. It employs the spatial correlation concept through the calculation of the correlation integral.
The correlation integral is computed as follows:

$$C_{\varepsilon,m} = \frac{1}{N_m(N_m - 1)} \sum_{i \neq j} I_{i,j;\varepsilon} \quad (1)$$

where, $I_{i,j} = 1$ if $\left\| x_i^m - x_j^m \right\| \leq \varepsilon$ and 0 otherwise.

N is the number of observations; m designates the embedding dimension and ε indicates the maximum distance between two pairs of points (i,j) with $1 \leq i \leq N$ and $1 \leq j \leq N$, in the m-dimensional space. If the time series is *i.i.d* we have:

$$C_{\varepsilon,m} \approx [C_{\varepsilon,1}]^m \quad (2)$$

The BDS statistic test $BDS_{\varepsilon,m} = \dfrac{\sqrt{N}[C_{\varepsilon,m} - (C_{\varepsilon,1})^m]}{\sqrt{V_{\varepsilon,m}}}$ (3)

With $\sqrt{V_{\varepsilon,m}}$ the standard deviation indicating that the series x_t is *i.i.d* if $N \rightarrow \infty$.

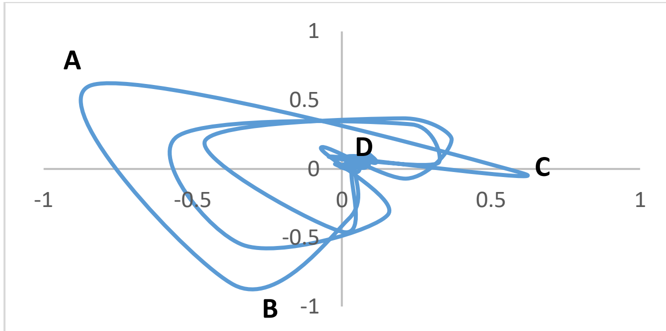
BDS test is a two-tailed test. We should reject the null hypothesis if the BDS test statistic is greater than the critical values (e.g. if α=0.05, the critical value = ±1.96).
For the Lebanese GDP growth rate, the results of this test are on table 1.

Table 1: Results of BDS test: The Lebanese GDP growth rate

Dimension	BDS Statistic	Std. Error	z-Statistic	Prob.
2	0.05	0.02	2.11	0.03
3	0.10	0.04	2.85	0.00
4	0.12	0.04	2.64	0.00
5	0.13	0.05	2.62	0.00

Table 1 indicates that for all dimensions the statistic BDS exhibits values greater than the critical value (since the probabilities are inferior to the 0.05 threshold). We can reject the null hypothesis of independence and identically distributed (*i.i.d*). This suggest that the Lebanese GDP growth rate is non-linearly dependent, which is a necessary but not sufficient condition of chaotic behavior.
In order to search chaos in the Lebanese GDP (Y_t), the phase portrait 2D (two dimensions) of GDP growth rate (dY_t) is built. Each ordered pair (dY_t, dY_{t-1}; t = 2, ..., N) is displayed in the plane (Figure 1) where x-axis represents the value of dY_{t-1} and y-axis, value of dY_t (Kriz, 2011).

Figure 1: GDP growth rate in the phase space



The individual points ($dY_{t,}$, dY_{t-1}) of the phase space are connected by a smooth curve. This curve seems to be a chaotic attractor as mentioned in the Lorenz model. In this Figure, the points farther from the "stable point" are related to times of greater economic uncertainty. For example, points A, B and C correspond to the war period and significant political instabilities (during the period 1975-1992) while point D represents a relative stability period (from 1995 to 2017).

3. The Lorenz model: The use of Lebanese macroeconomic data

Traditionally, chaos theory is analyzed by means of a logistic function used as a simple model of biological growth (Kemp, 1997) such as: $x_t = ax_{t-1}(1-x_t)$. From $a = 3.57$ to 4, the behavior of the variable x_t becomes chaotic. In our model, as the Baumol and Benhabib one, (1989, op. cit.), we search the occurrence of chaos in economic phenomenon like the evolution of the Lebanese GDP. For this, we use an empirical application of the Lorenz model (3.1). Such model leads to estimate the Lyapunov exponents (3.2) and represents the Lebanese GDP growth as a strange attractor generated by chaotic dynamic (3.3).

3.1. The Lorenz model: Presentation and empirical application

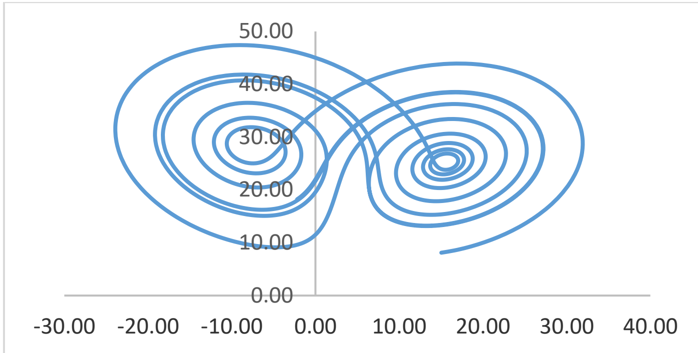
The Lorenz model is a famous example of chaotic three-dimensional system of differential equations. To search chaos in the Lebanese GDP, we use the Lorenz equations.
Lorenz has oversimplified a system of differential equations that would explain some of the unpredictable behavior of the weather (Hirsh, Smale, Devaney, 2004). The differential equations are the following:

$$x' = \sigma(y - x) \quad (4)$$
$$y' = rx - y - xz \quad (5)$$
$$z' = xy - bz \quad (6)$$

where' means the derivatives with respect to time ($x' = dx/dt$, for example).

Parameters σ and r are assumed to be positive and $\sigma > b + 1$. If $\sigma = 10$, $r = 28$ and $b = 8/3$ with initial conditions ($x = 10$, $y = 10$, $z = 10$), we obtain the Lorenz attractor represented by a strange figure drawing wings of a butterfly (Figure 2).

Figure 2: The Lorenz attractor: The wings of a butterfly



This figure in 2D (two dimensions) is issued from 1000 iterations with the values of coefficients of differential equations (4), (5) and (6).

We apply the three-dimensional system of differential equations by using data series of main macroeconomic aggregates regarding Lebanon over the period 1970-2017 (United Nations Statistic, 2019). All variables of the system are written in growth rate (i.e. in logarithm and first difference) to be stationary. These variables are GDP (*dY*), consumption expenditure (*dC*) and investment (*dI*).

However, before estimating the differential equations of the Lorenz model, we have to estimate the following relationship to show, on the one hand, the impact of consumption expenditure and investment on the GDP and on the other hand partial correlation between the three variables.

$$dY_t = a_1 dC_t + a_2 dI_t + \varepsilon_t \qquad (7)$$

Using OLS method, we obtain:

$$\underset{}{dY_t} = \underset{(11.86)^{***}}{0.60\,dC_t} + \underset{(8.37)^{***}}{0.28\,dI_t} + e_t \qquad (8)$$

$N = 47$; $R^2 = 0.94$, $e_t \rightarrow WN(0, \sigma_\varepsilon^2)$

N = observations; R^2 = the Coefficient of determination; (.) = the Student-ratio; *** = significance at the one-percent level.

The coefficients in this relationship are significant at the one-percent level. Residuals also follow a white noise (*WN*) process (Appendix 1) and the values of partial correlation show that the consumption expenditure is more correlated with GDP than investment (Appendix 2). Therefore, to determinate the order of the variables in the system, we choose the GDP as dependent variable in the first differential equation ($dY_t = x$), then consumption expenditure ($dC_t = y$) and finally investment ($dI_t = z$).

$$dY_t' = \sigma(dY_t + dC_t) + \varepsilon_t \qquad (9)$$
$$dC_t' = rdY_t - dC_t - dI_t dY_t + v_t \qquad (10)$$
$$dI_t' = dC_t dY_t - bdI_t + u_t \qquad (11)$$

By means of the seemingly unrelated regression (SUR) method taking into account the heteroscedasticity and contemporaneous correlation in the errors across equations, we obtain:

$$dY_t' = \underset{(4.54)^{***}}{0.299}(dY_t + dC_t) + e_t \qquad (12)$$

$$e_t = -\underset{(3.27)^{***}}{0.285}e_{t-1} + \zeta_{1t} \qquad (12')$$

$$dC_t' = \underset{(5.48)^{***}}{1.288}dY_t - dC_t - dI_t dY_t + v_t \qquad (13)$$

$$v_t = -\underset{(3.27)^{***}}{0.285}v_{t-1} + \zeta_{2t} \qquad (13')$$

$$dI_t' = dC_t dY_t - \underset{(7.10)^{***}}{0.845}dI_t + u_t \qquad (14)$$

$$u_t = -\underset{(3.27)^{***}}{0.285}u_{t-1} + \zeta_{3t} \qquad (14')$$

$N = 46$

N = observations; (.) indicates the Student-ratio; *** = significance at the one-percent level.

The coefficients in this differential equations system are significant at the one-percent level as well as the serial correlation coefficient common across equations. Thus, the Portmanteau test of residuals autocorrelation shows that there is no residuals autocorrelation in the system (Appendix 3).
The parameters σ, r and b are positive and $\sigma + 1 > b$. The coefficient of the GDP is greater than one (relation (13)). This means that solutions of the Lorenz system that start far from the origin do at least move closer in.

3.2. The Lyapunov exponents

From obtained values of differential equations (12), (13), (14), we estimate the characteristics of the GDP by simulating the Lorenz attractor. We take the values $\sigma = 0.299$; $r = 1.288$; $b = 0.845$ and calculate the eigenvalues of the linearized system in order to know the Lyapunov Exponents whose values, issued from the Jacobian matrix of linearized system, indicate if the attractor is reduced to:

- Stable fixe point: All the exponents are negative;
- Limit cycle: On exponent is zero and the remaining ones are all negative;
- Strange attractor generated by chaotic dynamic: At least one exponent is positive.

The linearized system is the following:

$$X' = \begin{bmatrix} \sigma & \sigma & 0 \\ r - dI & -1 & dY \\ dC & dY & -b \end{bmatrix} X = \begin{bmatrix} 0.299 & 0.299 & 0 \\ 1.288 - dI & -1 & dY \\ dC & dY & -0.845 \end{bmatrix} \mathrm{X} \tag{15}$$

With $dY = 0.088$; $dC = 0.088$ and $dI = 0.069$ as initial values. These values (by multiplying them by 100) indicate the growth rates of GDP, consumption expenditure and investment, respectively in 1970.

The eigenvalues of this system are issue from the following characteristic equation:

$$\mathrm{Det}(X' - \lambda I) = \begin{bmatrix} 0.299 - \lambda & 0.299 & 0 \\ 1.288 - dI & -1 - \lambda & dY \\ dC & dY & -0.845 - \lambda \end{bmatrix} = 0 \tag{16}$$

Expanding the determinant, we have:

$$\lambda^3 + a\,\lambda^2 + b\,\lambda + \mathrm{c} = 0 \tag{17}$$

By resolving this equation, we obtain the values of the Lyapunov exponents.
$\lambda_1 = 0.54$; $\lambda_2 = -0.83$; $\lambda_3 = -1.25$

We see that one exponent is positive. This means that the Lebanese GDP growth looks like a strange attractor generated by chaotic dynamic as the Figure 3-a and Figure 3-b plot it.

3.3. The Lebanese GDP growth as a strange attractor generated by chaotic dynamic

Figure 3-a and Figure 3-b show respectively GDP growth versus investment and consumption expenditure versus GDP growth.

Figure 3-a: GDP growth versus investment

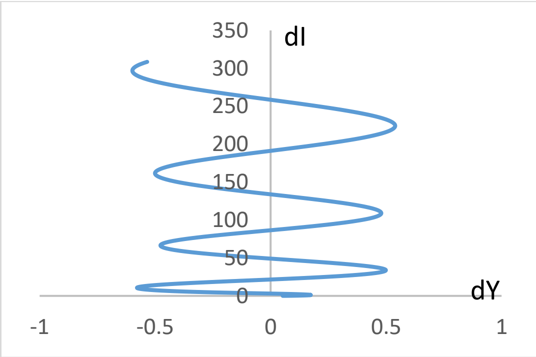


Figure 3-b: Consumption expenditure versus GDP growth

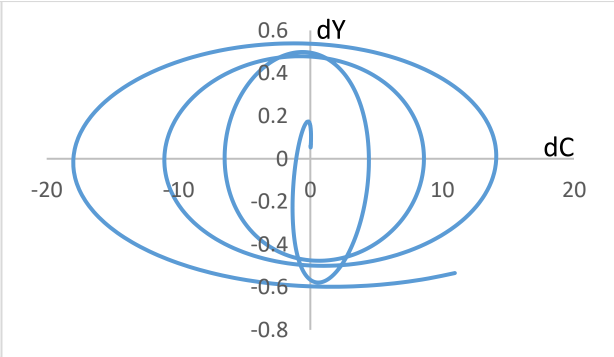


Figure 3-a and Figure 3-b in 2D are issued from 1000 iterations with the values of coefficients of differential equations (12), (13) and (14) and exhibit a strange attractor. The Figures show that the GDP presents a chaotic dynamic behavior that looks like an ellipsoid (notably in Figure 3-b where the partial correlation between GDP (dY_t) and consumption expenditure (dC_t) is the highest). Thus,

all solutions of the differential equations system that starting far from the origin are attracted to a set that sits inside the ellipsoid. (Hirsh, Smale, Devaney, 2004, p. 309). In fact, since $r > 1$ we can write:

$$V\,(dY, dC, dI) = rdY_t^2 + \sigma dC_t^{\,2} + \sigma\,(dI_t^2 - 2r)^2 \qquad (18)$$

Using the values of differential equations, we obtain: V = 1.288*0.088^2 + 0.299*0.088^2 + 0.299*(0.069 – 2*1.288)^2 = 1.89 = v

Thus, $v > 0$ defines an ellipsoid in R^3 centered.

We compute also the Lyapunov function:

$$\dot{V} = \frac{\partial V}{\partial dY}.dY' + \frac{\partial V}{\partial dC}.dC'\frac{\partial V}{\partial dI}.dI' \qquad (19)$$

By rearranging the terms we have:

$$\dot{V} = -2\sigma[rdY^2 + dC^2 + b(dI - r)^2 - br^2] \qquad (20)$$

With $rdY^2 + dC^2 + b(dI - r)^2 = \mu$ (21)

The equation (21) defines the ellipsoid when $\mu > 0$. Indeed, using the values of coefficients and initials values of macroeconomic variables in equation (21), we obtain: μ = 1.288*0.088^2 + 0.088^2 + 0.875*(0.069 – 1.288)^2 = 2.97.

The equation of ellipsoid can also be written as:

$$rdY^2 + dC^2 + b(dI - r)^2 = br^2 \qquad (22)$$

We have $\mu > br^2$ (with br^2 = 0.875*1.288^2 = 1.45) and consequently $\dot{V} < 0$.

All solutions that starts outside the ellipsoid eventually enters this ellipsoid and then remains trapped therein for all future time. This means that even though the Lebanese GDP exhibits chaotic behavior in the short run, a strange attractor does exist in the long run that pushes Lebanese GDP to regain regular growth. Indeed, the robustness of the Lebanese banking system (with assets close to four times the GDP) coupled with a strong private sector and high level of openness can partially explain this phenomenon. Moreover, the importance of the Lebanese diaspora (there are more Lebanese living outside Lebanon than within) plays a significant role because it allows Lebanon to benefit large financial inflows, reaching around 7.2 USD Billion in 2018 (World Bank, 2019).

4. Conclusion

The Lebanese GDP growth rate analyzed in the space phase presents a strange attractor generated by a chaotic dynamic. By estimating the differential equations of Lorenz system, solutions that start far from the origin return inside the ellipsoid. In economic terms, this means that the Lebanese GDP growth presents chaotic fluctuations in the short run. However, such GDP growth is attracted to its long run value after an exogenous shock for example. In fact, it seems that the Lebanese economic is somehow resilient even though the economic fluctuations in this country show a large magnitude especially during the period of civil war. However, nowadays, policy uncertainties and macroeconomic imbalances limit Lebanon's resilience as well as the Syrian crisis that has strained Lebanon's public finances and service deliver (Conde and Sanchez-Bella, 2018).

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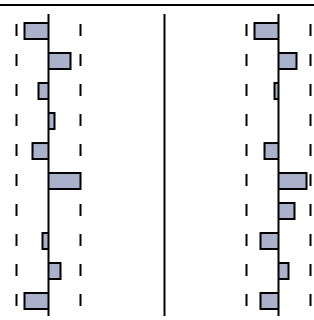
Appendix 1: White Noise test of relation (8)



Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.208	-0.208	2.1623	0.141
		2	0.201	0.165	4.2297	0.121
		3	-0.095	-0.028	4.7043	0.195
		4	0.052	-0.003	4.8475	0.303
		5	-0.140	-0.120	5.9294	0.313
		6	0.290	0.259	10.656	0.100
		7	0.002	0.150	10.657	0.154
		8	-0.052	-0.159	10.814	0.212
		9	0.105	0.082	11.481	0.244
		10	-0.208	-0.169	14.170	0.165

All probabilities corresponding to the Q-Stat statistic are greater than the 0.05 threshold. We cannot reject the null hypothesis that residuals of equation (8) follow a white noise process.

Appendix 2: Partial correlation between GDP growth rate (dY_t) consumption expenditure (dC_t) and Investment (dI_t).

	dY_t	dC_t	dI_t
dY_t	1.00		
dC_t	0.87	1.00	
dI_t	0.78	-0.46	1.00

The partial correlation between GDP growth rate (dY_t) and consumption expenditure (dC_t) is the highest.

Appendix 3: Residual Portmanteau Tests for Autocorrelations (equations (12), (13), (14))

Null Hypothesis: no residual autocorrelations up to lag h
Sample: 1973 2017
Included observations: 46

Lags	Q-Stat	Prob.	Adj Q-Stat	Prob.	df
1	13.75	0.13	14.06	0.12	9
2	22.11	0.23	22.81	0.20	18
3	26.52	0.49	27.53	0.43	27
4	35.27	0.50	37.14	0.42	36

The probabilities are greater to the 0.05 threshold for all lags and we cannot reject the null hypothesis of no residuals autocorrelation.