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Monopolistic competition with infinite product variety networks

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Abstract

In this article, I develop a new monopolistic competition model where the product variety space is a network defined as a "graphon". I determine and fully characterize the free-entry equilibrium of such a new setting. I obtain two major results. First, I find that denser networks generate less entries, lower aggregate quantities and lower welfare. Second, compared to the standard monopolistic competition model without network structure, the new setting generates more entries, higher aggregate quantities and higher welfare.

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1 Introduction

Network externalities have become inescapable in economics (see Jackson *et al.* 2017). Notably, network externalities play a central role in the pricing rule of firms (see Bloch 2016, and Ushchev and Zenou 2018). However, it is worth noting that this literature faces three drawbacks. First, the literature focuses solely on the case where there is a finite number of firms. Firms are either part of a duopoly or an oligopoly. The case with an infinite number of competitors is not studied. This assumption is restrictive because many markets are composed of a very large number of firms (see Aumann 1964). It is also restrictive because most theoretical models in macroeconomics and international trade are based on monopolistic competition with a continuum of firms. Second, the literature considers models with no entry. The free-entry case is left out. This means that the interaction between the consumer/product network and the decision to enter made by firms is not understood. Third, existing models in the literature are not very tractable. The main reason for this is that these models are based on the study of an adjacency matrix. Such a mathematical object is complicated to analyze and sometimes requires strong assumptions.[1] So, it is rare to derive explicit solutions, and models can be very cumbersome. This overall lack of tractability is undesirable as it translates into a lack of interpretability, making it difficult to identify the consequences of the network structure. As a consequence, extending the case from a finite number of firms to the case with an infinite number of firms should be more practical.

In other words, the effects of network externalities in a monopolistic competition model with free entry are still opened questions. Is it possible to develop a monopolistic competition model with infinite product variety networks? If so, how to model a network when the number of varieties is infinite? What are the features of a free-entry equilibrium?

In this article, I answer these questions. Toward that goal, I develop a new monopolistic competition model where preferences are linear-quadratic and where the degrees of substitutability of varieties are given by a graph/network ω. When there is an infinite number of varieties, the suitable notion of network is a graphon.[2] In particular, I emphasize that the best form for the graphon is when it is constant $\omega = \psi$ where ψ is a measure of the network topology. An increase in ψ (i.e., a denser network) means that products are more subtitutes. The constant graphon is the best form because it is standard, interpretable, tractable, finds empirical support and is microfounded since it is the unique limit of Erdós-Rényi random graphs.[3] I then determine the free-entry equilibrium (which exists and is

[1] For example, Ushchev and Zenou (2018) have to assume a profit function of the form $\pi = pq$ instead of the standard form $\pi = pq - cq - f$ to derive their elegant results.

[2] Graphon for graph function, see Glasscock (2016), Avella-Medina *et al.* (2018) and Parise and Ozdaglar (2018).

[3] See Lovász and Szegedy (2006) and Lovász (2012).

unique and explicit due to the constant graphon) of this new framework.

I find the following. I demonstrate that denser networks are associated with less entries, lower aggregate quantities and lower welfare. Equivalently, relative to the standard monopolistic competition model without network, the new model generates more entries, higher aggregate quantities and higher welfare. I offer an intuitive rationale for these results.

This article contributes to the literature of imperfect competition models with product variety networks. Within a large literature, Ushchev and Zenou (2018) are the first to characterize the Bertrand-Nash equilibrium of firms evolving in a finite graph. Chen *et al.* (2018) focus on a duopoly setting and show that the social network of consumers can affect the firms' prices. Chen *et al.* (2022) generalize Chen *et al.* (2018). They build an oligopolistic competition model with consumer networks and study how the topology network interacts with prices and the number of firms. To the best of my knowledge, the present article is the first to study a monopolistic competition model with infinite product variety networks and with free entry of firms. One contribution is to underline that the appropriate framework is the constant graphon when the number of varieties becomes infinite. Another contribution is to show that a free-entry equilibrium behaves well under the constant graphon, so that it is possible to identify all the effects of the network structure on key economic variables.

The article is structured in the following manner: Section 2 presents the model and the results, and Section 3 provides the conclusions.

2 Monopolistic competition and infinite networks

In what follows, I derive the results of the article step-by-step.

2.1 The constant graphon as the best model for infinite product networks

The economy is populated by a unit mass of identical households.

2.1.1 The Erdős-Rényi graph

Households derive utility U from consuming a set of N varieties of a differentiated good produced in a single industry. N is assumed to be (very) large meaning that the scale of the economy is nationwide. As usual, each variety is produced by a single firm.

As in Ottaviano *et al.* (2002) and Ushchev and Zenou (2018), U is linear-quadratic such that:

$$U\left(\mathbf{x}, \mathbf{G}^N\right) = x_0 + \alpha \sum_{i=1}^{N} x_i - \frac{\beta}{2} \sum_{i=1}^{N} x_i^2 - \gamma \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} x_i x_j$$

where $\mathbf{x}$ is the consumption profile, x_i is the consumption of variety $i \in \{1, ..., N\}$ and x_0 is the level of consumption of an outside good. α accounts for the consumers' willingness to pay for varieties, β captures the consumers' love for varieties and $\gamma \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} x_i x_j$ measures the substitutability linkages across varieties.

The interactions between varieties/firms are captured by an adjacency matrix denoted by $\mathbf{G}^N$. $\mathbf{G}^N = (g_{ij})_{i,j=1,\ldots,N}$ is an $N \times N$ symmetric matrix with binary entries $\{0, 1\}$: $g_{ij} = 1$ if there is a "link" between varieties i and j, $g_{ij} = 0$ otherwise, and $g_{ii} = 0$ for all $i \in \{1, ..., N\}$ and for all $j \in \{1, ..., N\}$. In other words, two varieties are "substituable" if they are connected.

In addition, as the number of firms is very large, the adjacency matrix has a specific form. The exact structure of the network does not matter: the exact distribution of "0" and "1" is inconsequential. The adjacency matrix can be randomly constructed, and what matters is the average interactions across varieties. In particular, the suitable notion for $\mathbf{G}^N$ is when $\mathbf{G}^N$ follows an Erdós-Rényi graph. Erdós-Rényi graphs constitute a (standard) class of finite graphs that are built by connecting nodes/varieties randomly. Any possible link between any two pair of nodes has a probability $\psi \in [0, 1]$ to exist.[4] When the number of firms is large, the adjacency matrix is an Erdós-Rényi graph for four reasons. First, it is well-acknowledged that Erdós-Rényi graphs are the best objects to generate, characterize and study large networks with binary entries (see Easley and Kleinberg 2010). Second, the interactions across varieties are controlled by a simple and interpretable parameter. ψ can be viewed as a global measure of network topology. An increase in ψ implies "denser" networks of firms as firms/varieties are more connected. Third, Erdós-Rényi graphs find empirical support in the sense that it is easy to estimate the value of ψ using data and using the algorithm proposed by Sealfon and Ullman (2019).[5] Last, Erdós-Rényi graphs have the best asymptotic properties among the class of random graphs (see next section).

2.1.2 The constant graphon

When $N \to +\infty$, households derive utility $\mathcal{U}$ such that:

$$\mathcal{U}(\mathbf{q}, \mathbf{W}) = q_0 + \alpha \int_0^1 q(i)di - \frac{\beta}{2} \int_0^1 q(i)^2 di - \gamma \int_0^1 \int_0^1 \omega(i,j) q(i) q(j) di dj$$

$\mathbf{q}$ is the consumption profile, $q(i)$ is the consumption of variety $i \in [0, 1]$ and q_0 is the level of consumption of an outside good.[6] The term $\gamma \int_0^1 \int_0^1 \omega(i,j) q(i) q(j) di dj$ summarizes the

[4] As a consequence, the probability that a firm has edges to $M \leq N$ other firms and no edges to the rest of $N - M$ firms is $\binom{N}{M} \psi^M (1-\psi)^{N-M}$. This also implies that the average number of connections in the graph is simply $N\psi$, and the associated variance is therefore $N(N-1)\psi^2$. Last, note that the product network is "complete" when $\psi = 1$ as all firms are direct competition with each other.

[5] See Appendix A for a short methodological comment.

[6] As before, α is the consumers' willingness to pay for varieties and β captures the consumers' love for varieties.

substitutability linkages across varieties, and $\mathbf{W}$ is the continuum analog of $\mathbf{G}^N$.

Here $\omega(i,j)$ is a graphon that describes the interaction between varieties i and j. A graphon is any symmetric $\mathcal{B}([0,1]) \times \mathcal{B}([0,1])$-measurable real-valued square-integrable function where $\mathcal{B}([0,1])$ is the Borel σ-field of the set $I = [0,1]$. ω is also connected to the graphon operator $\mathbf{W} : L^2([0,1]) \to L^2([0,1])$ defined by $[\mathbf{W}f]_i = \int_0^1 \omega(i,j)f(j)dj$ with $f \in L^2([0,1])$ and $i \in [0,1]$.

As $\mathbf{G}^N$ has a particular form, $\mathbf{W}$ and ω have a particular form too. Notably, it is well-acknowledged that, when varieties become infinite, a sequence of Erdós-Rényi random graphs denoted by $(\mathbf{G}^N)_{N\geq 1}$ converges (almost surely) toward the constant graphon $\omega(i,j) = \psi$ for all (i,j) in $[0,1]^2$ (see Lovász and Szegedy 2006, and Lovász 2012). It is worth noting that this convergence exists and is unique, explicit and tractable. In turn, this tractability implies that a free-entry equilibrium exists, is unique and has a closed-form solution (see Sections 2.2-2.3).[7]

Under this convergence, preferences can collapse to:

$$\mathcal{U}(\mathbf{q}, \mathbf{W}) = q_0 + \alpha \int_0^1 q(i)di - \frac{\beta}{2}\int_0^1 q(i)^2 di - \gamma\psi \int_0^1 \int_0^1 q(i)q(j)didj$$

Remind that γ captures the degree of subtitutability between any two substitutes and that ψ is the probability that any two varieties are substitutes. Consequently, $\gamma\psi$ can be interpreted to be the average degree of substitutability between firms. Equivalently, $\gamma\psi$ represents a measure of the extent/toughness of competition between firms.

2.2 Infinite-firm game with no entry

2.2.1 Consumers' program

The budget constraint of households is as follows: $q_0 + \int_0^1 p(i)q(i)di = \overline{q}_0 + y$ where $\overline{q}_0$ is the initial endowment in the numéraire, $p(i)$ is the price of variety $i \in [0,1]$ and y is the revenue of households. Under this environment, the consumers' program is: $\max_{\mathbf{q}} \quad \mathcal{U}(\mathbf{q}, \mathbf{W}) \quad s.t. \quad q_0 + \int_0^1 p(i)q(i)di = \overline{q}_0 + y$. Solving this problem leads to:

$$p(i) = \alpha - \beta q(i) - \gamma\psi \int_0^1 q(j)dj, \quad \forall i \in [0,1]$$

This is the standard inverse demand function augmented by the presence of the constant graphon ψ.[8]

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[7] See Appendix B for a summary of all the desirable properties of the constant graphon.

[8] Set $\psi = 1$ to find the classical inverse demand function in standard monopolistic competition models.

2.2.2 Firms' program

Plugging the expression of $p(i)$ into the traditional definition of the profit function π yields:

$$\pi(i) = p(i)q(i) - cq(i) - f = \left[\alpha - \beta q(i) - \gamma\psi \int_0^1 q(j)dj\right] q(i) - cq(i) - f, \quad \forall i \in [0,1]$$

where $c > 0$ is the constant marginal cost and $f > 0$ is the fixed cost. Consequently, the firm i's program is as follows: $\max_{q(i)\geq 0} \left\{\left[\alpha - \beta q(i) - \gamma\psi \int_0^1 q(j)dj\right] q(i) - cq(i) - f\right\}$.

2.2.3 Symmetric equilibrium

Under this setup, a symmetric equilibrium denoted by $(q^*_{no}, p^*_{no}, \mathcal{U}^*_{no})$ is as follows.

Proposition 1 *A symmetric equilibrium is characterized by the following:*[9]

$$\frac{\partial q^*_{no}}{\partial \psi} < 0, \quad \frac{\partial p^*_{no}}{\partial \psi} < 0, \quad \frac{\partial \mathcal{U}^*_{no}}{\partial \psi} < 0.$$

and

$$q^*_{no} = \frac{\alpha - c}{2\beta + \gamma\psi} \geq \tilde{q}_{no}, \quad p^*_{no} = \frac{\alpha\beta}{2\beta + \gamma\psi} + \frac{(\beta + \gamma\psi)c}{2\beta + \gamma\psi} \geq \tilde{p}_{no}, \quad \mathcal{U}^*_{no} \geq \tilde{\mathcal{U}}_{no}.$$

where the superscript $\tilde{\bullet}_{no}$ *denotes the result in the traditional monopolistic competition model with no entry but without network.*

The effect of network topology is unambiguous. An increase in ψ induces lower prices, lower quantities and lower welfare. Equivalently, the new monopolistic model predicts higher prices, higher quantities and higher welfare relative to the standard monopolistic model. In both cases, the explanation is the same.[10] By definition, a denser network means that varieties are more often linked and so substitutes. In practice, this means that each firm produces a less differentiated product. As a result, firms are now operating in a tougher competitive environment. They know that if they raise their prices, part of their demand will shift to other competitors, since goods have become more similar/substitutable. This squeezes their margins and puts downward pressure on prices. In addition, the fact that goods are more homogeneous also means that more firms are addressing the same demand. Firms serve a smaller market. At equilibrium, their market share declines. The combination of lower prices and lower quantities also leads to lower profitability. Last, at first examination, the welfare consequences of an increase in the density of the product-variety network are unclear. Lower prices benefit consumers, while lower quantities penalize them. Here, the net result is negative, since the negative effect of lower quantities always outweighs the positive effect of lower prices.

[9] See Appendix C for a proof.

[10] It is easy to demonstrate that $\frac{\partial q^*_{no}}{\partial \psi} < 0 \Rightarrow q^*_{no} > \tilde{q}_{no}$, $\frac{\partial p^*_{no}}{\partial \psi} < 0 \Rightarrow p^*_{no} > \tilde{p}_{no}$ and $\frac{\partial \mathcal{U}^*_{no}}{\partial \psi} < 0 \Rightarrow \mathcal{U}^*_{no} > \tilde{\mathcal{U}}_{no}$ as long as $\psi < 1$. Remind that $\psi < 1$ implies that the role of competition in the new framework is diminished compared to the standard model.

2.3 Infinite-firm game with free entry

So far, the number of firms is fixed. Hereafter, I relax this assumption by considering that there are $n > 0$ (potential) competitors in the industry.

2.3.1 Consumers' program

In that case, the linear-quadratic preferences reduce to the following:

$$\mathcal{U}(\mathbf{q}, \mathbf{W}) = q_0 + \alpha \int_0^n q(i)di - \frac{\beta}{2} \int_0^n q(i)^2 di - \gamma\psi \int_0^n \int_0^n q(i)q(j)didj$$

Solving the consumers' program leads to:

$$p(i) = \alpha - \beta q(i) - \gamma\psi \int_0^n q(j)dj, \quad \forall i \in [0, n]$$

Under free entry, the degree of competition faced by firms is measured by $\gamma\psi n$ where $\gamma\psi$ is the overall degree of product substitutability and n is the number of competitors.

2.3.2 Firms' program

Plugging the expression of $p(i)$ into the profit function π yields:[11]

$$\pi(i) = \left[\alpha - \beta q(i) - \gamma \int_0^n \psi q(j)dj\right] q(i) - cq(i) - f, \quad \forall i \in [0, n]$$

and the firm i's program is: $\max_{q(i)\geq 0} \left\{\left[\alpha - \beta q(i) - \gamma\psi \int_0^n q(j)dj\right] q(i) - cq(i) - f\right\}$.

2.3.3 Symmetric equilibrium

Under this new environment, a symmetric equilibrium denoted by $(n_e^*, q_e^*, p_e^*, Q_e^*, \mathcal{U}_e^*)$ is as follows.

Proposition 2 *A symmetric equilibrium is characterized by the following:*[12]

$$\frac{\partial n_e^*}{\partial \psi} < 0, \quad \frac{\partial q_e^*}{\partial \psi} = 0, \quad \frac{\partial p_e^*}{\partial \psi} = 0, \quad \frac{\partial Q_e^*}{\partial \psi} < 0, \quad \frac{\mathcal{U}_e^*}{\partial \psi} < 0.$$

and

$$n_e^* = \frac{1}{\gamma\psi} \left(\frac{\alpha - c}{\sqrt{\frac{f}{\beta}}} - 2\beta \right) \geq \tilde{n}_e, \quad q_e^* = \tilde{q}_e, \quad p_e^* = \tilde{p}_e, \quad Q_e^* \geq \tilde{Q}_e, \quad \mathcal{U}_e^* \geq \tilde{\mathcal{U}}_e.$$

where $Q = nq$ is global quantity and where the superscript $\tilde{\bullet}_e$ indicates the result in the traditional monopolistic competition model with free entry but without network.

[11] In this article, firms are assumed to be homogeneous. However, it is possible to consider heterogeneous firms with different cost functions. This may complicate the model and I leave this for future research.

[12] See Appendix D for a proof.

As in Proposition 1, the effect of network topology is clear. An increase in ψ, implying an increase in variety substitutability, results in lower firm entry (i.e. extensive margin), lower aggregate quantities and lower welfare. However, a denser network has no impact on quantities per firm (intensive margin) and prices. Equivalently, the new model predicts more firms on the market, higher overall quantities and higher welfare than the classical model. In both cases, the intuition is the same.[13] Since firms now produce less differentiated goods, they compete more fiercely for consumers. This squeezes their margins and lowers their profit expectations. In turn, this limits the entry of new actors into the market.[14] Unlike the case without endogenous entries, producing more similar goods is inconsequential on prices and the intensive margin of firms. This is a standard property of linear-quadratic preferences. Intuitively, the fact that each firm sells a more substitutable good should lower prices and quantities for the reasons outlined in Section 2.2. What changes under the free-entry condition is that the toughness of competition between firms depends on both the general substitutability of varieties captured by $\gamma\psi$ and the number of active firms n. A denser network induces more substitutability, but also fewer competitors. The net effect is therefore generally ambivalent. Under linear-quadratic preferences, the two opposing effects cancel each other out, leaving individual quantities and prices unchanged. In addition, the fact that the intensive margin remains the same implies that the global quantity Q^* and welfare $\mathcal{U}^*$ are entirely pinned down by the extensive margin. Since there are fewer firms on the market, aggregate quantities decrease. Last, as each firm produces a specific variety, fewer firms mean fewer varieties of the good, and therefore less choice for consumers. This naturally reduces consumer welfare.

3 Conclusions

In this article, I develop a new monopolistic competition model where preferences are linear-quadratic and where the product variety space is a network defined as a graphon. I study the free-entry equilibrium of such a framework. I demonstrate that denser networks are associated with less entries, lower aggregate quantities and lower welfare. Relative to the standard monopolistic competition model without network structure, the new setting generates more entries, higher aggregate quantities and higher welfare.

[13] It is easy to demonstrate that $\frac{\partial n_e^*}{\partial \psi} < 0 \Rightarrow n_e^* > \tilde{n}_e$, $\frac{\partial q_e^*}{\partial \psi} = 0 \Rightarrow q_e^* = \tilde{q}_e$, $\frac{\partial p_e^*}{\partial \psi} = 0 \Rightarrow p_e^* = \tilde{p}_e$, $\frac{\partial Q_e^*}{\partial \psi} < 0 \Rightarrow Q_e^* > \tilde{Q}_e$ and $\frac{\partial \mathcal{U}_e^*}{\partial \psi} < 0 \Rightarrow \mathcal{U}_e^* > \tilde{\mathcal{U}}_e$ as long as $\psi < 1$. Remind that $\psi < 1$ implies that the role of competition in the new framework is diminished compared to the standard model.

[14] Remind that ψ is a parameter that controls competition toughness. If ψ increases, varieties are more often linked and so substitutes.

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A Methodological comment

It is worth noting that adopting a statistical point of view when the number of competitors becomes large is fairly natural in economics. This is the famous result of Aumann (1964) stating that exchange economies converge toward markets with an atomless continuum of traders. What matters is the statistical distribution of traders when the number of these traders becomes large. This point of view is also present in Game Theory. For example, in an $n-$player game, it is common to substitute the strategy profile of the other players a_{-i} by the empirical measure $\sum_{k\neq i}^{n-1}\delta_{a_k}$ of the other agents such that $r(a_i, a_{-i}) = r(a_i, \sum_{k\neq i}^{n-1}\delta_{a_k})$ where r is the reward of the game, a_i is the action/strategy of player i and δ is the Dirac probability measure.

B Why is the constant graph the best network model when the number of varieties is infinite?

The constant graph is the best network model when the number of varieties is infinite, because the constant graph is:

- standard, since the constant graphon is a basic graphon in the literature.
- interpretable, since the ψ parameter is a measure of the substitutability of the good's varieties.
- finds empirical support, since the ψ parameter can be easily estimated.
- simple and therefore tractable. This tractability is key, since it means that a free-entry equilibrium exists, is unique and has a closed-form expression. The fact that the free-entry equilibrium is explicit makes it possible to determine the effects of network structure on quantities, prices, profits, the number of varieties and consumer welfare.
- microfounded since the constant graphon is the limit of Erdós-Rényi graphs, and this limit is well posed because it exists and is unique.

In turn, Erdós-Rényi graphs have desirable properties as they are:

- standard, since they are basic random graphs in the literature.
- intuitive, since they only assumes that any two pair of nodes has a probability ψ to exist
- finds empirical support, since the ψ parameter can be easily estimated.

To my best knowledge, only the constant graphon has all these good properties. To confirm this, note that another explicit graphon is the power graphon: $\omega(i, j) = (ij)^{\psi}$ with $\psi > 0$ being a constant. Unfortunately, such a graphon emerges as the limit of coupled Kuramoto phase oscillators, which has no economic sense. Moreover, this graphon is not tractable as the free-entry condition becomes the following:

$$\int_0^1 \left(\left[\left(\mathbf{I} + \frac{\gamma n}{2\beta} \mathbf{W} \right)^{-1} \mathbf{1} \right]_i \right)^2 di = \left(\frac{2\beta}{\alpha - c} \right)^2 \sqrt{\frac{f}{\beta}}$$

with

$$\left[\left(\mathbf{I} + \frac{\gamma n}{2\beta} \mathbf{W} \right)^{-1} \mathbf{1} \right]_i = 1 + \frac{\frac{\gamma n}{2\beta}(1 - 2\gamma)}{(1-\gamma)\left(1 - 2\gamma - \frac{\gamma n}{2\beta}\right) i^{\gamma}}$$

This makes the free-entry condition highly non-linear such that the equilibrium mass of firms may not exist and may not be unique.

C Proof of Proposition 1

The firm i's program is defined as follows:

$$\max_{q(i) \geq 0} \left\{ \left[\alpha - \beta q(i) - \gamma \int_0^1 \omega(i,j) q(j) dj \right] q(i) - c q(i) - f \right\}$$

Such a problem is "well-posed" in the sense that there exists a unique solution denoted by $q^*_{no}(i)$ and that is determined by the following first order condition:

$$q^*_{no}(i) = \frac{\alpha - c}{2\beta} - \frac{\gamma}{2\beta} \int_0^1 \omega(i,j) q^*_{no}(j) dj, \quad i \in [0,1]$$

Then, using the property of $\mathbf{W}$, and defining the resolvent operator as follows:

$$\left(\mathbf{I} + \frac{\gamma}{2\beta} \mathbf{W} \right)^{-1} = \mathbf{I} + \frac{\gamma}{2\beta} \mathbf{W} - \left(\frac{\gamma}{2\beta} \right)^2 \mathbf{W}^2 + \left(\frac{\gamma}{2\beta} \right)^3 \mathbf{W}^3 - ...$$

the vector of equilibrium quantities denoted by $\mathbf{q}^*_{no}$ can be expressed as:[15]

$$\mathbf{q}^*_{no} = \frac{\alpha - c}{2\beta} \left(\mathbf{I} + \frac{\gamma}{2\beta} \mathbf{W} \right)^{-1} \mathbf{1}$$

As the limiting graphon from Erdós-Rényi graphs is the constant graphon, it can be readily verified that $\mathbf{W}$ satisfies: $\mathbf{W1} = \psi^k \mathbf{1}$ with k being a positive integer. As a consequence, the resolvent operator becomes:

$$\left(\mathbf{I} + \frac{\gamma}{2\beta} \mathbf{W} \right)^{-1} \mathbf{1} = \mathbf{1} + \frac{\gamma\psi}{2\beta} \mathbf{1} - \left(\frac{\gamma\psi}{2\beta} \right)^2 \mathbf{1}^2 + \left(\frac{\gamma\psi}{2\beta} \right)^3 \mathbf{1}^3 - ... = \frac{2\beta}{2\beta + \gamma\psi} \mathbf{1}$$

[15]Existence and uniqueness of $\mathbf{q}^*$ are established if $\frac{\gamma}{2\beta} \|\mathbf{W}\| < 1$ where $\|\|$ denotes the operator norm. The operator norm coincides with the largest eigenvalue of $\mathbf{W}$ (see Avella-Medina et al. (2018) and Parise and Ozdaglar (2018) for more details).

and the equilibrium quantity/consumption *per* firm denoted by q_{no}^* is:

$$q_{no}^* = \frac{\alpha - c}{2\beta + \gamma\psi} \geq \tilde{q}_{no} = \frac{\alpha - c}{2\beta + \gamma}, \quad \frac{\partial q_{no}^*}{\partial \psi} < 0$$

as $\psi \leq 1$, and where $\tilde{q}_{no}$ is the equilibrium quantity in the standard monopolistic competition model with no entry but with no network. Integrating this relationship into the pricing rule gives the equilibrium price *per* firm denoted by p_{no}^* such that:

$$p_{no}^* = \frac{\alpha\beta}{2\beta + \gamma\psi} + \frac{(\beta + \gamma\psi)c}{2\beta + \gamma\psi} \geq \tilde{p}_{no} = \frac{\alpha\beta}{2\beta + \gamma} + \frac{(\beta + \gamma)c}{2\beta + \gamma}, \quad \frac{\partial p_{no}^*}{\partial \psi} < 0$$

as $\psi \leq 1$, and where $\tilde{p}_{no}$ is the equilibrium price in the standard monopolistic competition model with no entry but with no network. Last, note that the following holds in equilibrium:

$$\begin{aligned}
\mathcal{U}\left(q_{no}^*\right) &= q_0 + \alpha q_{no}^* - \frac{\beta}{2}\left(q_{no}^*\right)^2 - \gamma\psi\left(q_{no}^*\right)^2 \\
&= q_0 + \left(\alpha - \beta q_{no}^* - \gamma\psi q_{no}^*\right) q_{no}^* + \frac{\beta}{2} q_{no}^* \\
&= q_0 + p_{no}^* q_{no}^* + \frac{\beta}{2} q_{no}^*
\end{aligned}$$

As $\psi \leq 1$, $\frac{\partial q_{no}^*}{\partial \psi} < 0$ and $\frac{\partial p_{no}^*}{\partial \psi} < 0$, this implies that:

$$\frac{\partial \mathcal{U}_{no}^*}{\partial \psi} < 0 \quad \text{and} \quad \mathcal{U}_{no}^* \geq \tilde{\mathcal{U}}_{no}$$

where $\tilde{\mathcal{U}}_{no}$ is the equilibrium utility in the standard monopolistic competition model with no entry but with no network.

D Proof of Proposition 2

The firm i's program is defined as follows:

$$\max_{q(i) \geq 0} \left\{ \left[\alpha - \beta q(i) - \gamma\psi \int_0^n q(j)dj \right] q(i) - cq(i) - f \right\}$$

Such a problem is "well-posed" in the sense that there exists a unique solution denoted by $q_e^*(i)$ and that is determined by the following first order condition:

$$q_e^*(i) = \frac{\alpha - c}{2\beta} - \frac{\gamma\psi}{2\beta} \int_0^n q_e^*(j)dj, \quad \forall i \in [0, n]$$

Solving the above equation yields:

$$q_e^* = \frac{\alpha - c}{2\beta + \gamma\psi n}$$

In addition, integrating the expression of q_e^* into the profit function yields:

$$\pi = \beta\left(\frac{\alpha - c}{2\beta + \gamma\psi n}\right)^2 - f$$

Assuming free entry $\pi = 0$, and after some algebra, I find the following equilibrium mass of firms denoted by n_e^*:

$$n_e^* = \frac{1}{\gamma\psi}\left(\frac{\alpha - c}{\sqrt{\frac{f}{\beta}}} - 2\beta\right) \geq \tilde{n}_e = \frac{1}{\gamma}\left(\frac{\alpha - c}{\sqrt{\frac{f}{\beta}}} - 2\beta\right), \quad \frac{\partial n_e^*}{\partial\psi} < 0$$

as $\psi \leq 1$, and where $\tilde{n}_e$ is the equilibrium mass of firms in the standard monopolistic competition model without network. Plugging this solution into the expression q_e^* leads to:

$$q_e^* = \sqrt{\frac{\beta}{f}} = \tilde{q}_e, \quad \frac{\partial q_e^*}{\partial\psi} = 0$$

where $\tilde{q}_e$ is the equilibrium individual quantity in the standard monopolistic competition model without network. Using the pricing rule, this also implies that:

$$p_e^* = \alpha - \beta\sqrt{\frac{\beta}{f}} - \left(\frac{\alpha - c}{\sqrt{\frac{f}{\beta}}} - 2\beta\right) = \tilde{p}_e, \quad \frac{\partial p_e^*}{\partial\psi} = 0$$

where $\tilde{p}_e$ is the equilibrium price in the standard monopolistic competition model without network. Similarly, defining aggregate quantities as $Q = nq$, it is readily verified that:

$$Q_e^* = \frac{1}{\gamma\psi}\left(\frac{\alpha - c}{\sqrt{\frac{f}{\beta}}} - 2\beta\right)\sqrt{\frac{\beta}{f}} \geq \tilde{Q}_e = \frac{1}{\gamma}\left(\frac{\alpha - c}{\sqrt{\frac{f}{\beta}}} - 2\beta\right)\sqrt{\frac{\beta}{f}}, \quad \frac{\partial Q_e^*}{\partial\psi} < 0$$

as $\psi \leq 1$. Last, note that the following holds in equilibrium:

$$\begin{aligned}
\mathcal{U}(q_e^*) &= q_0 + \alpha n_e^* q_e^* - \frac{\beta}{2} n_e^* (q_e^*)^2 - \gamma\psi (n_e^* q_e^*)^2 \\
&= q_0 + (\alpha - \beta q_e^* - \gamma\psi n_e^* q_e^*) n_e^* q_e^* + \frac{\beta}{2} n_e^* q_e^* \\
&= q_0 + p_e^* n_e^* q_e^* + \frac{\beta}{2} n_e^* q_e^*
\end{aligned}$$

As $\psi \leq 1$, $p_e^* = \tilde{p}_e$ and $Q_e^* \geq \tilde{Q}_e$, this means that:

$$\mathcal{U}_e^* \geq \tilde{\mathcal{U}}_e$$

where $\tilde{\mathcal{U}}_e$ is the equilibrium utility in the standard monopolistic competition model with no network. Similarly, as $\frac{\partial p_e^*}{\partial\psi} = 0$ and $\frac{\partial Q_e^*}{\partial\psi} < 0$, this implies that:

$$\frac{\mathcal{U}_e^*}{\partial\psi} < 0$$

E Finite-firm game

E.1 Consumers' program

Households face the following budget constraint: $x_0 + \sum_{i=1}^{N} p_i x_i = \overline{x}_0 + y$ with p_i being the price of variety i, $\overline{x}_0$ being the endowment of x_0 and y being the revenue of households. Under this environment, the consumers' program is defined as: $\max_{\mathbf{x}} \quad U\left(\mathbf{x}, \mathbf{G}^N\right) \quad s.t. \quad x_0 + \sum_{i=1}^{N} p_i x_i = \overline{x}_0 + y$. Solving the consumers' program gives:

$$p_i = \alpha - \beta x_i - \gamma \sum_{j=1}^{N} g_{ij} x_j$$

This is the standard inverse demand function augmented by the linkages across varieties. Set $g_{ij} = 1$ for all (i, j) to end up with the standard result in monopolistic competition models. As U is linear-quadratic, the inverse demand function is independent of y the revenue of households.

E.2 Firms' program and equilibrium quantities/consumptions

Integrating the above result into the profit function Π gives:

$$\Pi_i = p_i x_i - c x_i - f = \left(\alpha - \beta x_i - \gamma \sum_{j=1}^{N} g_{ij} x_j\right) x_i - c x_i - f$$

where c is the constant marginal cost and f is the fixed cost. In that case, the firm i's program is simply: $\max_{x_i} \left\{\left(\alpha - \beta x_i - \gamma \sum_{j=1}^{N} g_{ij} x_j\right) x_i - c x_i - f\right\}$. Solving the firm i's program leads to the following:

$$x_i = \frac{\alpha - c}{2\beta} - \frac{\gamma}{2\beta} \sum_{j=1}^{N} g_{ij} x_j$$

and equilibrium quantities denoted by $\mathbf{x}^*$ can be re-written in "matrix form" as follows:[16]

$$\mathbf{x}^* = \frac{\alpha - c}{2\beta} \left(\mathbf{I}_N + \frac{\gamma}{2\beta} \mathbf{G}^N\right)^{-1} \mathbf{1}_N$$

$\mathbf{I}_N$ is the N identity matrix and $\mathbf{1}_N$ is the N-dimensional vector of ones and $\left(\mathbf{I}_N + \frac{\gamma}{2\beta} \mathbf{G}^N\right)^{-1}$ is defined as follows:

$$\left(\mathbf{I}_N + \frac{\gamma}{2\beta} \mathbf{G}^N\right)^{-1} = \mathbf{I}_N + \frac{\gamma}{2\beta} \mathbf{G}^N - \left(\frac{\gamma}{2\beta}\right)^2 \mathbf{G}^{N2} + \left(\frac{\gamma}{2\beta}\right)^3 \mathbf{G}^{N3} - ...$$

[16] Following the Perron-Frobenius theorem, the equation is well-defined if $\frac{\gamma}{2\beta} < \frac{1}{\lambda_{max}(\mathbf{G}^N)}$ with $\lambda_{max}(\mathbf{G}^N)$ being the largest eigenvalue of $\mathbf{G}^N$.

In line with Ushchev and Zenou (2018), equilibrium quantities are a function of centrality measures as $\mathbf{x}^*$ depends on the sign-alternating Bonacich centrality measures of varieties (see Bonacich (1987)). Last, note that, as the limiting graphon from Erdós-Rényi graphs is the constant graphon, it can be readily verified that the following holds:

$$\mathbf{x}^* \to \mathbf{q}^* \quad when \quad N \to \infty$$

F Discussions

So far, the analysis of the model has been focused on Industrial Economics. However, the model may have marketing and management implications. These implications could be the subject of future, more in-depth research. With regard to marketing, a denser product network should encourage each firm to invest more in marketing to counter the rise in competition by making the product more differentiated. In this case, the γ parameter becomes the firm's choice, and an increase in ψ should imply a decrease in γ in the medium term. With regard to management, it is possible to assume that the productive capacity of firms depends positively on management effort. In this case, the effect of an increase in competition induced by a denser/connected consumer network could be lessened by an increase in management effort.

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