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Majority judgment and majority criterion

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Abstract

Majority criterion says that an alternative should not be chosen if there is another alternative which majority of voters prefer to it. It is well known that this criterion is too strong: there is no social choice rule that satisfies it when there are at least three alternatives. In this paper, we show that majority judgment, reformulated as a social choice rule, satisfies a weaker variant of majority criterion, referred to as shuffling majority criterion. In addition, we show that if a social choice rule satisfies this axiom and another one concerning non-manipulability, it should pick only those alternatives with the highest "median grade".

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1 Introduction

In voting situations, each voter submits his or her ranking over alternatives. Given a list of such individuals rankings, a social choice rule chooses an alternative or a set of alternatives. In rating situations, each evaluator assigns a grade to each alternative, and a rating rule aggregates those individual evaluations. In "elective situations", each evaluator also assigns a grade to each alternative. Given a list of such individual evaluations, a (generalized) social choice rule chooses a set of alternatives.

Our purpose is to show *majority judgment*, which is a rating rule devised by Balinski and Laraki (2007), satisfies a certain desirable property when it is reformulated as a social choice rule in elective situations. Majority judgment is well-known for its excellence as a rating rule. As shown in Balinski and Laraki (2011), majority judgment is immune to strategic manipulation in the sense that each voter cannot make the rating of an alternative closer to the one he or she wants. However, as we shall see, when it is reformulated as a social choice rule in elective situations, even majority judgment cannot escape from Gibbard-Satterthwaite Theorem. Nevertheless, we demonstrate that there is an advantage to using majority judgment in elective situations.

We justify majority judgment in terms of a principle embodied in majority rule. *Majority criterion* (May 1952) has been recognized as an important principle representing the idea of democratic decision making. However, in general, there is no social choice rule that satisfies this criterion when there are at least three alternatives. In this paper, we introduce a weaker variant of majority criterion, referred to as *shuffling majority criterion*, and show that majority judgment satisfies this requirement.

This paper is organized as follows. In the next section, we set up the model. Then, we introduce *shuffling majority criterion* and confirm the majority judgment rule satisfies this property. Finally, we show that only social choice rules that pick up alternatives with the highest median grade can satisfy *shuffling majority criterion* and *nonmanipulability by other evaluations*, a property about independence between alternatives.

2 Model

There are a finite set of alternatives $X := \{x_1, x_2, \ldots, x_m\}$ and an odd-numbered set of individuals $N := \{1, 2, \ldots, n\}$. Let G be a *grade set*, and $\succ_G$ be an associated strict ordering on G. We assume that there are potentially countless grade sets. Let $\mathcal{G}$ be the family of grade sets. We make the following assumption between two grade sets which have inclusion relationship:

Consistency among grade sets. For all $G, H \in \mathcal{G}$ with $G \subseteq H$ and all $g, g' \in G$,

$$g \succeq_G g' \iff g \succeq_H g'.$$

For all $G \in \mathcal{G}$ and all $i \in N$, i's list of *evaluations* is

$$R_i = \begin{pmatrix} r(x_1, i) \\ r(x_2, i) \\ \vdots \\ r(x_m, i) \end{pmatrix} \in G^m.$$

For all $R \in \bigcup_{G \in \mathcal{G}} G^{m \times n}$ and all $x \in X$, we write

$$R^x := (r(x,1),\ r(x,2),\ \ldots,\ r(x,n)) \in G^n.$$

For all $G \in \mathcal{G}$, all $g \in G^n$, and all $\ell \in \{1,2,\ldots,n\}$, let $\rho(g,\ell)$ be the ℓth "smallest" grade in terms of $\succ_G$.[1] A *rating rule* is a function which maps each profile of lists of individual evaluations to a list of social evaluation. For each alternative $x \in X$, *majority judgment* (Balinski and Laraki 2011), as a rating rule, picks the "median grade", $\rho\left(R^x, \frac{n+1}{2}\right)$.

A *social choice rule* $F\colon \bigcup_{G \in \mathcal{G}} G^{m \times n} \to 2^X \setminus \{\varnothing\}$ is a function which maps each profiles of lists of individual evaluations to a nonempty subset of alternatives. We are particularly interested in the following social choice rules which are based on majority judgment:

Majority judgment rule, F_{MJ}. For all $R \in \bigcup_{G \in \mathcal{G}} G^{m \times n}$ and all $x \in X$, $x \in F_{MJ}(R)$ if and only if, for all $y \in X$,
$\left(\rho\left(R^x, \frac{n+1}{2}\right), \rho\left(R^x, \frac{n+1}{2}-1\right), \rho\left(R^x, \frac{n+1}{2}+1\right), \rho\left(R^x, \frac{n+1}{2}-2\right), \rho\left(R^x, \frac{n+1}{2}+2\right), \ldots, \rho\left(R^x, 1\right), \rho\left(R^x, n\right)\right)$
is lexicographically greater than or equal to
$\left(\rho\left(R^y, \frac{n+1}{2}\right), \rho\left(R^y, \frac{n+1}{2}-1\right), \rho\left(R^y, \frac{n+1}{2}+1\right), \rho\left(R^y, \frac{n+1}{2}-2\right), \rho\left(R^y, \frac{n+1}{2}+2\right), \ldots, \rho\left(R^y, 1\right), \rho\left(R^y, n\right)\right)$;[2]

Majority grade rule, F_{MG}. For all $R \in \bigcup_{G \in \mathcal{G}} G^{m \times n}$ and all $x \in X$, $x \in F_{MJ}(R)$ if and only if, for all $y \in X$, $\rho\left(R^x, \frac{n+1}{2}\right) \succeq \rho\left(R^y, \frac{n+1}{2}\right)$.

Clearly, the outcome of F_{MJ} is always contained in the one of F_{MG}:

Fact 1. For all $R \in \bigcup_{G \in \mathcal{G}} G^{m \times n}$, $F_{MJ}(R) \subseteq F_{MG}(R)$.

3 Shuffling majority criterion

Consider the following requirement for a social choice function introduced by May (1952).

Majority criterion. For all $x, y \in X$ and all $R \in \bigcup_{G \in \mathcal{G}} G^{m \times n}$,
if $|\{i \in N \mid r(x,i) \prec r(y,i)\}| > |\{i \in N \mid r(x,i) \succ r(y,i)\}|$, then $x \notin F(R)$.

In general, there is no social choice rule satisfying *majority criterion*.

Proposition 1. Assume that $n \geqq 3$ and $m \geqq 3$. Then, there is no social choice rule satisfying *majority criterion*.

Proof. Let $N = \{1,2,3\}$, $X = \{x,y,z\}$, and $G = \{Good, Fair, Bad\}$ with $Good \succ_G Fair \succ_G Bad$. Let

$$R := \begin{pmatrix} r(x,1) & r(x,2) & r(x,3) \\ r(y,1) & r(y,2) & r(y,3) \\ r(z,1) & r(z,2) & r(z,3) \end{pmatrix} = \begin{pmatrix} Bad & Fair & Good \\ Good & Bad & Fair \\ Fair & Good & Bad \end{pmatrix} \in G^{3 \times 3}.$$

If F satisfies *majority criterion*, then $x \notin F(R)$, $y \notin F(R)$, and $z \notin F(R)$. Thus, $F(R) = \varnothing$, which contradicts the nonempty-valuedness of F. □

[1] Namely, for all $G \in \mathcal{G}$, all $g = (g_1, g_2, \ldots, g_n) \in G^n$, and all $\ell \in \{1,2,\ldots,n\}$, it is satisfied that
- $\rho(g,\ell) \in \{g_1, g_2 \ldots, g_n\}$, and
- $|\{i \in N \mid g_i \prec \rho(g,\ell)\}| < \ell \leqq |\{i \in N \mid g_i \preceq \rho(g,\ell)\}|$.

[2] For all $G \in \mathcal{G}$ and all $g, g' \in G^n$, g is *lexicographically greater* than g' if $\{i \in N \,|\, g_i \neq g'_i\} \neq \varnothing$ and $g_j \succ g'_j$ where $j = \min\{i \in N \,|\, g_i \neq g'_i\}$.

Note that Proposition 1 is a straight forward application of the Condorcet paradox. We offer a weaker condition of *majority criterion*, namely, *shuffling majority criterion*. As an illustration of *shuffling majority criterion*, consider an election to select a few from among several alternatives, to be conducted in a polling station. There are as many ballot boxes as alternatives. Each voter has a graded evaluation (e.g., the three-grade scale of *Good*, *Fair*, and *Bad*) to each alternative, and puts a piece of paper (unsigned) with his or her evaluation to each alternative into each ballot box. In such a situation, after everyone has voted, majority criterion cannot be applied because each evaluation is anonymous. Given two alternatives, say x and y, in order to apply majority criterion, we need to count the number of voters whose evaluation to x is higher than his or her evaluation to y. However, in the above situation, such information is lost. Basically, *shuffling majority criterion* takes all possible cases into account: this evaluation could be his or hers, and so on. It says that if majority of voters evaluate x to be higher than y in all possible cases, then y should not be elected.

For an arbitrary set A, $\Pi(A)$ denotes the set of bijections from A into itself.

Shuffling majority criterion. For all $x, y \in X$ and all $R \in \bigcup_{G \in \mathcal{G}} G^{m \times n}$ if, for all $\pi \in \Pi(N)$, $|\{i \in N \mid r(x,i) \prec r(y,\pi(i))\}| > |\{i \in N \mid r(x,i) \succ r(y,\pi(i))\}|$, then $x \notin F(R)$.

Fact 2. If a social choice function satisfies *majority criterion*, then it satisfies *shuffling majority criterion*.

Proposition 2. F_{MJ} satisfies *shuffling majority criterion*.

Proof. Let $x, y \in X$ and $R \in \bigcup_{G \in \mathcal{G}} G^{m \times n}$. Assume that, for all $\pi \in \Pi(N)$, $|\{i \in N \mid r(x,i) \prec r(y,\pi(i))\}| > |\{i \in N \mid r(x,i) \succ r(y,\pi(i))\}|$. Suppose, by contradiction, that $x \in F_{MJ}(R)$. By the definition of F_{MJ}, $\left(\rho\left(R^x, \frac{n+1}{2}\right), \rho\left(R^x, \frac{n+1}{2}-1\right), \rho\left(R^x, \frac{n+1}{2}+1\right), \rho\left(R^x, \frac{n+1}{2}-2\right), \rho\left(R^x, \frac{n+1}{2}+2\right), \ldots, \rho\left(R^x, 1\right), \rho\left(R^x, n\right)\right)$ is lexicographically greater than or equal to $\left(\rho\left(R^y, \frac{n+1}{2}\right), \rho\left(R^y, \frac{n+1}{2}-1\right), \rho\left(R^y, \frac{n+1}{2}+1\right), \rho\left(R^y, \frac{n+1}{2}-2\right), \rho\left(R^y, \frac{n+1}{2}+2\right), \ldots, \rho\left(R^y, 1\right), \rho\left(R^y, n\right)\right)$. Let $A = (a_1, a_2, \ldots, a_n)$ be the former vector, and $B = (b_1, b_2, \ldots, b_n)$ the latter. If $A = B$, clearly the assumption is violated. Therefore, there exists $j \in \{1, 2, \ldots, n\}$ such that $a_j \succ b_j$ and $a_i = b_i$ for all $i < j$.

[Case 1] j is odd.

Then, the following three conditions are satisfied.

- $a_1 = b_1$, $a_2 = b_2$, ..., and $a_{j-1} = b_{j-1}$,
- $a_j \succ b_j$, and
- $a_{j+2} \succ b_{j+1}$, $a_{j+4} \succ b_{j+3}$, ..., and $a_n \succ b_{n-1}$.

Note that the third condition holds because $a_n \succeq \cdots \succeq a_{j+4} \succeq a_{j+2} \succeq a_j \succ b_j \succeq b_{j+1} \succeq b_{j+3} \succeq \cdots \succeq b_{n-1}$. Therefore, there exists $\pi' \in \Pi(N)$ such that $|\{i \in N \mid r(x,i) \succ r(y,\pi'(i))\}| \geqq \frac{n-(j+2)}{2} + 1 + 1 = \frac{n-j}{2} + 1$ and $|\{i \in N \mid r(x,i) = r(y,\pi'(i))\}| \geqq j - 1$.
This implies that $|\{i \in N \mid r(x,i) \prec r(y,\pi'(i))\}| \leqq n - \left(\frac{n-j}{2} + 1\right) - (j-1) = \frac{n-j}{2} - 2$. Hence, $|\{i \in N \mid r(x,i) \prec r(y,\pi'(i))\}| < |\{i \in N \mid r(x,i) \succ r(y,\pi'(i))\}|$. This contradicts the assumption.

[Case 2] j is even.

Then, the following three conditions are satisfied.

- $a_1 = b_1$, $a_2 = b_2$, ..., and $a_{j-1} = b_{j-1}$,
- $a_j \succ b_j$, and
- $a_{j+3} \succ b_{j+2}$, $a_{j+5} \succ b_{j+4}$, ..., and $a_n \succ b_{n-1}$.

Note that the third condition holds because $a_n \succeq \cdots \succeq a_{j+5} \succeq a_{j+3} \succeq a_j \succ b_j \succeq b_{j+2} \succeq b_{j+4} \succeq \cdots \succeq b_{n-1}$. Therefore, there exists $\pi' \in \Pi(N)$ such that $|\{i \in N \mid r(x,i) \succ r(y,\pi'(i))\}| \geqq \frac{n-(j+3)}{2} + 1 + 1 = \frac{n-j-1}{2} + 1$ and $|\{i \in N \mid r(x,i) = r(y,\pi'(i))\}| \geqq j-1$.
This implies that $|\{i \in N \mid r(x,i) \prec r(y,\pi'(i))\}| \leqq n - \left(\frac{n-j-1}{2}+1\right) - (j-1) = \frac{n-j-1}{2} - 1$. Hence, $|\{i \in N \mid r(x,i) \prec r(y,\pi'(i))\}| < |\{i \in N \mid r(x,i) \succ r(y,\pi'(i))\}|$. This contradicts the assumption. □

4 Axiomatic analysis

Suppose that, in an elective situation, y is elected, but x is not. Then, *nonmanipulability by other evaluations* says that if individuals change their evaluations of alternatives other than x and y, x remains not to be elected (even if the grade set becomes smaller).

Nonmanipulability by other evaluations. For all $x, y \in X$, all $G, H \in \mathcal{G}$ with $G \subseteq H$, all $R \in G^{m\times n}$, and all $R' \in H^{m\times n}$ such that for all $i \in N$, $r'(x,i) = r(x,i)$ and $r'(y,i) = r(y,i)$,

$$y \in F(R') \text{ and } x \notin F(R') \implies x \notin F(R).$$

Now we show that if a social choice rule satisfies *shuffling majority criterion* and *nonmanipulability by other evaluations*, then it should pick up those alternatives with the highest median grade. As a corollary, we find that a social choice rule which satisfies *neutrality*, *anonymity*, *shuffling majority criterion*, and *nonmanipulability by other evaluations* is a subcorrespondence of the majority grade rule.

Theorem 1. Let $m \geqq 3$. If a social choice rule F satisfies *shuffling majority criterion* and *nonmanipulability by other evaluations*, then, for all $x, y \in X$, all $G \in \mathcal{G}$, and all $R \in G^{m\times n}$,

$$\rho\left(R^x, \frac{n+1}{2}\right) \prec_G \rho\left(R^y, \frac{n+1}{2}\right) \implies x \notin F(R).$$

Proof. Assume that F satisfies *shuffling majority criterion* and *nonmanipulability by other evaluations*. Let $x, y \in X$, $G \in \mathcal{G}$, and $R \in G^{m\times n}$ such that $\rho\left(R^x, \frac{n+1}{2}\right) \prec_G \rho\left(R^y, \frac{n+1}{2}\right)$. Note that $m \geqq 3$. Let $H \in \mathcal{G}$ be another grade set satisfying

- $H = G \cup \{h\}$ with $h \notin G$, and
- $\rho\left(R^x, \frac{n+1}{2}\right) \prec_H h \prec_H \rho\left(R^y, \frac{n+1}{2}\right)$.

Let $w \in X \setminus \{x, y\}$. Consider $R' \in H^{m\times n}$ satisfying

- for any $z \in X \setminus \{w\}$, $R'^z = R^z$, and
- $R'^w = (h, h, \cdots, h)$.

Since $R'^x = R^x$ and $\rho\left(R^x, \frac{n+1}{2}\right) \prec_H h$, $\rho\left(R'^x, \frac{n+1}{2}\right) \prec_H h$. Therefore, $\rho(R'^x, 1) \preceq_H \rho(R'^x, 2) \preceq_H \cdots \preceq_H \rho\left(R'^x, \frac{n+1}{2}\right) \prec_H h$. This means that, for all $\pi \in \Pi(N)$, $|\{i \in N \mid r'(x,i) \prec r'(w,\pi(i))\}| > |\{i \in N \mid r'(x,i) \succ r'(w,\pi(i))\}|$. By *shuffling majority criterion*, $x \notin F(R')$.

On the other hand, since $R'^y = R^y$ and $h \prec_H \rho\left(R^y, \frac{n+1}{2}\right)$, $h \prec_H \rho\left(R'^y, \frac{n+1}{2}\right)$. Therefore, $h \prec_H \rho\left(R'^y, \frac{n+1}{2}\right) \preceq_H \rho\left(R'^y, \frac{n+2}{2}\right) \preceq_H \cdots \preceq_H \rho(R'^y, n)$. This means that, for all $\pi \in \Pi(N)$, $|\{i \in N \mid r'(w,i) \prec r'(y,\pi(i))\}| > |\{i \in N \mid r'(w,i) \succ r'(y,\pi(i))\}|$. By *shuffling majority criterion*, $w \notin F(R')$. By the nonempty-valuedness of F, there exists $v \in F(R')$. By *nonmanipulability by other evaluations*, $x \notin F(R)$. □

Corollary 1. Let $m \geqq 3$. If a social choice rule F satisfies *shuffling majority criterion* and *nonmanipulability by other evaluations*, then, for all $R \in \bigcup_{G\in\mathcal{G}} G^{m\times n}$, $F(R) \subseteq F_{MG}(R)$. Moreover, F_{MG} satisfies the two properties.

Finally, we check the independence of *shuffling majority criterion* and *nonmanipulability by other evaluations*.

Example 1. Consider the following social choice rule $\hat{F}$: for all $R \in \bigcup_{G\in\mathcal{G}} G^{m\times n}$ and all $w \in X$,

$$w \in \hat{F}(R) \iff \begin{array}{l} w \in F_{MG}(R), \text{ or} \\ \text{for all } v \in X, \text{ there exists } \pi \in \Pi(N) \text{ such that} \\ \left|\{i \in N \mid r(v,i) \prec r(w,\pi(i))\}\right| \geqq \left|\{i \in N \mid r(v,i) \succ r(w,\pi(i))\}\right|. \end{array}$$

Then, $\hat{F}$ obviously satisfies *shuffling majority criterion*. Let $N = \{1,2,3\}$, $X = \{x,y,z\}$, and $G = \{Very\,Good, Good, Fair, Bad, Very\,Bad\}$ with $Very\,Good \succ_G Good \succ_G Fair \succ_G Bad \succ_G Very\,Bad$. Let

$$R := \begin{pmatrix} R^x \\ R^y \\ R^z \end{pmatrix} = \begin{pmatrix} Bad & Fair & Good \\ Very\,Bad & Good & Very\,Good \\ Very\,Bad & Very\,Bad & Very\,Bad \end{pmatrix} \in G^{3\times 3},$$

and $R' \in G^{3\times 3}$ with $R'^x = R^x$, $R'^y = R^y$, and $R'^z = (Good\ \ Good\ \ Good)$. Then, $\hat{F}(R) = \{x,y\}$ and $\hat{F}(R') = \{y,z\}$. Since $y \in \hat{F}(R')$, $x \notin \hat{F}(R')$, and $x \in \hat{F}(R)$, $\hat{F}$ violates *nonmanipulability by other evaluations*. Note that $\rho\left(R^x, \frac{n+1}{2}\right) \prec_G \rho\left(R^y, \frac{n+1}{2}\right)$ and $x \in \hat{F}(R)$. Thus, *nonmanipulability by other evaluations* is necessary for Theorem 1.

Example 2. Consider the following social choice rule $\tilde{F}$: for all $R \in \bigcup_{G\in\mathcal{G}} G^{m\times n}$ and all $w \in X$,

$$w \in \tilde{F}(R) \iff \text{for all } v \in X,\ \rho(R^w, n) \succeq \rho(R^v, n).$$

Then, $\tilde{F}$ obviously satisfies *nonmanipulability by other evaluations*. Let $N = \{1,2,3\}$, $X = \{x,y,z\}$, and $G = \{Very\,Good, Good, Fair, Bad, Very\,Bad\}$ with $Very\,Good \succ_G Good \succ_G Fair \succ_G Bad \succ_G Very\,Bad$. Let

$$R := \begin{pmatrix} R^x \\ R^y \\ R^z \end{pmatrix} = \begin{pmatrix} Bad & Bad & Very\,Good \\ Fair & Good & Good \\ Fair & Fair & Fair \end{pmatrix} \in G^{3\times 3}.$$

Then, $\tilde{F}(R) = \{x\}$. However, *shuffling majority criterion* requires that $x \notin \tilde{F}(R)$. Hence, $\tilde{F}$ violates *shuffling majority criterion*. Note that $\rho\left(R^x, \frac{n+1}{2}\right) \prec_G \rho\left(R^y, \frac{n+1}{2}\right)$ and $x \in \tilde{F}(R)$. Thus, *shuffling majority criterion* is also necessary for Theorem 1.

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