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How rare are egalitarian binary relations?

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Abstract

Knoblauch (2014) and Knoblauch (2015) investigate the relative size of the collection of binary relations with desirable features as compared to the set of all binary relations using symmetric difference metric (Cantor) topology and Hausdorff metric topology. We consider Ellentuck and doughnut topologies to further this line of investigation. We report the differences among the size of the useful binary relations in Cantor, Ellentuck and doughnut topologies.

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1 Introduction

This paper contributes to the research agenda introduced in recent papers, Knoblauch (2014) and Knoblauch (2015), which deals with the study of some context free features of binary relations. In this paper we consider the symmetric difference metric topology on the set $S := \{0,1\}^{\mathbb{N}}$ from Knoblauch (2014) and call it the *Cantor topology*.[1] We assume the set of alternatives X to be countably infinite which implies that the cardinality of the set $X \times X$ is also countably infinite. This allows us to describe any arbitrary binary relation via a coding set which is an element of $S := \{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$. This slight variation of the definition is useful to then approach two other topological concepts introduced in descriptive set theory of the real numbers: the Ellentuck and doughnuts topology.

The result (based on symmetric difference metric topology) in Knoblauch (2014) could be re-phrased as follows. The coding set of binary relations satisfying some well-known *basic* properties (e.g., transitivity, completeness, asymmetry, antisymmetry, linearity) are nowhere dense subsets of $2^{\mathbb{N}}$ with respect to Cantor topology and they are null sets with respect to Lebesgue measure. Relying on the coding set, we use Ellentuck and doughnut topology to further explore the rarity of the binary relations satisfying the basic properties.

The second class of binary relations examined in this paper are those satisfying equity or efficiency properties from the social choice literature. These principles are well-studied in the economic literature, for example in Diamond (1965), Hammond (1976), d'Aspremont and Gevers (1977). We retain the cardinality of $X \times X$ as countably infinite and show in Proposition 3 that the binary relations satisfying each of these axioms are not rare in doughnut topology whereas they are rare in Ellentuck (and similarly in Cantor) topology.

The paper is organized as follows. We introduce notation and the definitions in section 2. The binary relations satisfying basic properties are examined in section 3. Section 4 deals with the binary relations satisfying equity or efficiency axioms. We conclude in section 5.

2 Preliminaries

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ the set of real numbers. For all $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

2.1 Coding set

Let X be a countably infinite set of alternatives (utility streams) and let $\{x_n : n \in \mathbb{N}\}$ be the enumeration of all elements in X. Let $\{q_k : k \in \mathbb{N}\}$ enumerate all pairs in $X \times X$. A binary relations $\Re$ on X can be then coded/seen as a subset of $\mathbb{N}$ by collecting those indices $k \in \mathbb{N}$ for which the corresponding pair $q_k \in \Re$. Moreover, by standard identification of a subset with its characteristic function one can think of $\{0,1\}^{\mathbb{N}}$ (also denoted by $2^{\mathbb{N}}$)[2] as the set of all codes of binary relations on X. We denote the binary sequence in $2^{\mathbb{N}}$ coding the

[1] It is well-known and straightforward to check that Cantor topology and symmetric different topology coincide.

[2] We use both the notations interchangeably throughout the paper.

binary relation $\Re \subseteq X \times X$ by $z_{\Re}$. The notation $\Re_z$ is used for the binary relation coded by $z \in 2^{\mathbb{N}}$. Following examples would be helpful to clarify the notations.

Let the set $X = \{x_1, x_2, x_3, \cdots\}$ contain countably infinitely many elements. The number of all possible pairs of elements in set $X \times X$ is countably infinite as well, with the enumeration denoted by q_k, $k \in \mathbb{N}$. We fix this enumeration of the pairs and define binary relations as a sequence $z \in \{0,1\}^{\mathbb{N}}$. Thus, $z = \{1, 1, \cdots\}$ describes a binary relation containing all pairs of alternatives, i.e., $\Re_z = \{(x_i, x_j) : x_i, x_j \in X \ \forall i, j \in \mathbb{N}\}$. If $q_1 = (x_1, x_3)$, then $z' = \{0, 1, 1, \cdots\}$ describes a binary relation containing all pairs of alternatives except (x_1, x_3) i.e., $\Re_{z'} = \Re_z \setminus (x_1, x_3)$.

2.2 Ideals of small subsets

The notion of *small* subsets of $2^{\mathbb{N}}$ is captured by the following notion.

Definition. Ideal $\mathcal{I}_U$ of *U-small subsets* of $2^{\mathbb{N}}$: Let U be a non-empty collection of subsets of $2^{\mathbb{N}}$ (i.e. $U \subseteq \mathcal{P}(2^{\mathbb{N}})$) such that:

- for all $u \in U$ there exists non-empty $u' \subseteq u$ such that $u' \in U$, and
- for all $x \in 2^{\mathbb{N}}$ there exists $u \in U$ such that $x \in u$.

Set $X \in \mathcal{I}_U$ if and only if for every $u \in U$ there exists non-empty $u' \in U$, $u' \subseteq u$ such that $u' \cap X = \emptyset$.

For instance, the notion of U-small subsets generalizes the concepts of nowhere dense and Lebesgue null sets. Indeed if U is the collection of all open sets with respect to the Cantor topology, then $\mathcal{I}_U$ is exactly the ideal of nowhere dense sets with respect to the Cantor topology. It is easy to check that $\mathcal{I}_U$ is the ideal of Lebesgue measure zero sets when we consider U to be the collection of all closed subsets of $2^{\mathbb{N}}$ with positive Lebesgue measure.

Our objective is to analyze various notions of *smallness* of a collection of subsets of a set so as to extend the investigation on the *rarity* of properties of binary relations. We employ two notions of smallness (borrowed from the descriptive set theory), namely the ideal of *Ramsey* null sets (also called *Ellentuck* nowhere dense sets) and the ideal of *doughnut* null sets. Following definitions are needed in order to capture these notions.

2.3 Cantor, Ellentuck and doughnut collections

A *partial function* $f : X \to Y$ is a function from a subset S of X to $Y \subset \mathbb{R}$. If S equals X, the partial function is said to be total. Domain and range of function f are denoted by $\mathsf{dom}(f)$ and $\mathsf{ran}(f)$ respectively.

Definition. *Let $f : \mathbb{N} \to \{0,1\}$ be a partial function and define $N_f := \{x \in 2^{\mathbb{N}} : \forall n \in \mathsf{dom}(f)(x(n) = f(n))\}$.*

Cantor *collection γ: It consists of all N_f such that $\mathsf{dom}(f)$ is finite.*

Ellentuck *collection* ε*: It consists of all* N_f *such that* $\textsf{dom}(f)$ *and* $\mathbb{N} \setminus \textsf{dom}(f)$ *are both infinite and there exists* $k \in \mathbb{N}$ *for all* $n \in \textsf{dom}(f)(n \geq k \Rightarrow f(n) = 0)$.[3]

Doughnut *collection* δ*: It consists of all* N_f *such that* $\textsf{dom}(f)$ *and* $\mathbb{N} \setminus \textsf{dom}(f)$ *are both infinite.*

The sets in γ, ε and δ are the basic open sets of the Cantor, Ellentuck and doughnut topologies respectively. Ellentuck and Doughnuts topologies are well-studied in descriptive set theory.

2.4 Equity and Pareto principles

We will be dealing with following equity and Pareto principles in this paper. The anonymity (also called finite anonymity) axiom is a notion of procedural equity. Strong equity belongs to the class of consequentialist equity. The efficiency notion we use is the standard Pareto principle. Definitions and formal notations are as below.

Definition. *Let* $\Re \subseteq X \times X$ *be a binary relation.*

Anonymity: $\Re$ *is said to be* anonymous *if and only if for all* $\mathbf{t}, \mathbf{t}' \in X = Y^N$ *there are* $i, j \leq N$, $\mathbf{t}(j) = \mathbf{t}'(i)$ *and* $\mathbf{t}(i) = \mathbf{t}'(j)$ *and for all* $k \neq i, j$, $\mathbf{t}(k) = \mathbf{t}'(k)$, *then* $(\mathbf{t}', \mathbf{t}) \in \Re$ *and* $(\mathbf{t}, \mathbf{t}') \in \Re$ *hold, i.e.,* ($\mathbf{t} \sim_a \mathbf{t}'$).

$$\mathbf{t} \sim_a \mathbf{t}' \Leftrightarrow \exists i, j \leq N(\mathbf{t}(j) = \mathbf{t}'(i) \wedge \mathbf{t}(i) = \mathbf{t}'(j) \wedge \forall k \neq i, j(\mathbf{t}(k) = \mathbf{t}'(k))).$$

Strong equity: $\Re$ *is said to satisfy* strong equity *if and only if for all* $\mathbf{t}, \mathbf{t}' \in Y^N$ *there exist* $i, j \leq N$ *such that* $\mathbf{t}(i) < \mathbf{t}'(i) < \mathbf{t}'(j) < \mathbf{t}(j)$ *and for all* $k \neq i, j$, $\mathbf{t}(k) = \mathbf{t}'(k)$, *then* $(\mathbf{t}', \mathbf{t}) \in \Re$ *and* $(\mathbf{t}, \mathbf{t}') \notin \Re$, *i.e.,* ($\mathbf{t} <_s \mathbf{t}'$).

$$\mathbf{t} <_s \mathbf{t}' \Leftrightarrow \exists i, j \leq N(\mathbf{t}(i) < \mathbf{t}'(i) < \mathbf{t}'(j) < \mathbf{t}(j)) \wedge \forall k \neq i, j(\mathbf{t}(k) = \mathbf{t}'(k)).$$

Pareto principle: $\Re$ *is said to be* Paretian *if and only if for all* $\mathbf{t}, \mathbf{t}' \in Y^N$ *for all* $i \leq N$, $\mathbf{t}(i) \leq \mathbf{t}'(i)$ *and there exists* $i \leq N$ *such that* $\mathbf{t}(i) < \mathbf{t}'(i)$, *then* $(\mathbf{t}', \mathbf{t}) \in \Re$ *and* $(\mathbf{t}, \mathbf{t}') \notin \Re$, *i.e.,* ($\mathbf{t} <_p \mathbf{t}'$)

$$\mathbf{t} <_p \mathbf{t}' \Leftrightarrow \forall i \leq N(\mathbf{t}(i) \leq \mathbf{t}'(i)) \wedge \exists i \leq N(\mathbf{t}(i) < \mathbf{t}'(i)).$$

3 Basic properties of binary relations

In economic theory, social welfare relations and preference relations satisfy some properties, which we may a priori split into two categories: *basic* properties, such as reflexivity, irreflexivity, symmetry, asymmetry and transitivity; *economic* property, such as Pareto, anonymity, strong equity. In this section we consider the basic properties of binary relations, in the next one we deal with the economic properties.

[3]See Jech (2003, p. 524, Definition 25.26) for a textbook definition of the Ellentuck topology. Brendle et al. (2005) defines and investigates the properties of doughnut topology.

Proposition 1. *Let* **T**, **IR**, **A**, *and* **AS** $\subseteq 2^{\mathbb{N}}$ *be the coding sets of all transitive, irreflexive, asymmetric and antisymmetric binary relations respectively. Then there exists* N_f, $N_{f'}$, N_g, $N_h \in \varepsilon$ *such that* $N_f \subseteq \mathbf{T}$, $N_{f'} \subseteq \mathbf{IR}$, $N_g \subseteq \mathbf{A}$ *and* $N_h \subseteq \mathbf{AS}$. *In particular,* **T**, **IR**, **A**, *and* **AS** *are not* ε*-small.*

Proof. We give a detailed proof for transitivity and leave to the reader the other cases.

It is sufficient to prove that there exists $N_f \in \varepsilon$ such that every $x \in N_f$ codes a transitive binary relation on X.

Let

$$R := \{n_k \in \mathbb{N} : \text{ there exists } k \in \mathbb{N}, \text{ such that } q_{n_k} = (x_k, x_k)\}. \tag{1}$$

Set R is both infinite and co-infinite and collects the enumeration of all pairs in $X \times X$ with identical elements chosen in the pair (i.e., the reflexive part of any binary relation). Note that for all $n \in \mathbb{N} \setminus R$, $q_n = (x_j, x_m)$ with $x_j \neq x_m$. Next, let

$$n_1 \in \min\{n : n \in \mathbb{N} \setminus R\} \text{ with } q_{n_1} = (x_{j_1}, x_{m_1}), \text{ and} \tag{2}$$

$$n'_1 \in (\mathbb{N} \setminus R) \setminus \{n_1\}, \text{ such that } q_{n'_1} = (x_{m_1}, x_{j_1}). \tag{3}$$

Having defined (n_1, n'_1),$\cdots$, (n_{k-1}, n'_{k-1}), let

$$n_k \in \min\{n : n \in (\mathbb{N} \setminus R) \setminus \{n_1, n'_1, \cdots, n_{k-1}, n'_{k-1}\} \text{ with } q_{n_k} = (x_{j_k}, x_{m_k}), \text{ and} \tag{4}$$

$$n'_k \in (\mathbb{N} \setminus R) \setminus \{n_1, n'_1, \cdots, n_{k-1}, n'_{k-1}, n_k\}, \text{ such that } q_{n'_k} = (x_{m_k}, x_{j_k}). \tag{5}$$

Denote the disjoint subsets of $\mathbb{N}$ recursively defined in (4) and (5) as A and B respectively, i.e.,

$$A := \{n_k : k \in \mathbb{N}\}, \text{ and } B := \{n'_k : k \in \mathbb{N}\}. \tag{6}$$

The set $\mathbb{N} \setminus R$ has been partitioned in the sets A and B in the following manner. First (minimum) and each subsequent element of B lists the same pair of distinct elements but in reverse order as the first (minimum) and the corresponding subsequent element of A. Also let

$$\Gamma := \{(n_k, n'_k) : n_k \in A, \text{ and } n'_k \in B, k \in \mathbb{N}\}. \tag{7}$$

Set Γ is a sequential listing of elements in $A \times B$.

Let $\mathsf{dom}(f) = A \cup B$ and $f : \mathbb{N} \to 2$ be such that for all $n \in \mathsf{dom}(f)$, $f(n) = 0$. To show that every element in N_f codes a transitive binary relation, pick $z \in N_f$ arbitrarily and let $\Re_z$ be the corresponding binary relation. Recall that transitivity is the following property:

$$\text{For all } x, x', x'' \in X, ((x, x') \in \Re_z \wedge (x', x'') \in \Re_z \Rightarrow (x, x'') \in \Re_z).$$

As a consequence, $\Re_z$ trivially satisfies it because the left hand side of the implication never holds, since all pairs in $\Re_z$ are of the form (x_n, x_n), unless $x = x' = x''$ in which case the property trivially holds. Therefore $N_f \subseteq \mathbf{T}$. □

Remark 1. Proposition 1 distinguishes Ellentuck topology from Cantor topology. Knoblauch (2014) has shown that the binary relations satisfying basic properties are rare in Cantor topology. In contrast, the transitive, asymmetric or antisymmetric binary relations are not rare in Ellentuck topology.

Proposition 2. *Let* $\mathbf{C}$*,* $\mathbf{S}$*,* $\mathbf{R} \subseteq 2^{\mathbb{N}}$ *be the coding sets of all complete, symmetric and reflexive binary relations, respectively. Then* $\mathbf{C}$*,* $\mathbf{S}$*,* $\mathbf{R}$ *are* ε*-small, but they contain open subsets in* δ*, and so in particular,* $\mathbf{C}$*,* $\mathbf{S}$*,* $\mathbf{R}$ *are not* δ*-small.*

Proof. We give details proofs for complete relations and leave to the reader the other cases.

$\mathbf{C}$ is ε-small: Pick arbitrarily $N_f \in \varepsilon$, we aim to find $N_g \in \varepsilon$, $N_g \subseteq N_f$ such that $N_g \cap \mathbf{C} = \emptyset$. Let

$$\mathsf{dom}'(f) := \{n \in \mathsf{dom}(f) : f(n) = 0\}. \tag{8}$$

Observe that $\mathsf{dom}'(f)$ is an infinite set. For $k \in \mathsf{dom}'(f)$, there are $j_k, m_k \in \mathbb{N}$ such that $q_k = (x_{j_k}, x_{m_k})$. Also there is $k' \in \mathbb{N}$ such that $q_{k'} = (x_{m_k}, x_{j_k})$. Pick $k \in \mathsf{dom}'(f)$ such that $k' > k$. There are two cases.

(1) $k' \in \mathsf{dom}(f)$: Then $N_f \cap \mathbf{C} = \emptyset$, for both $f(k) = f(k') = 0$ and so neither (x_j, x_m) nor (x_m, x_j) are in any binary relation coded by any $z \in N_f$.

(2) $k' \notin \mathsf{dom}(f)$: Then the partial function $g : \mathbb{N} \to 2$ with $\mathsf{dom}(g) := \mathsf{dom}(f) \cup \{k'\}$ is defined as:

$$g(n) := \begin{cases} f(n) & \text{if } n \in \mathsf{dom}(f) \\ 0 & \text{if } n = k'. \end{cases} \tag{9}$$

Note that N_g is a well-defined subset in ε and $N_g \subseteq N_f$. Moreover, since $g(k) = g(k') = 0$, it follows that neither (x_j, x_m) nor (x_m, x_j) are in any binary relation coded by any $z \in N_g$; which gives $N_g \cap \mathbf{C} = \emptyset$.

$\mathbf{C}$ is not δ-small: We need to show that there is a set $N_f \in \delta$ such that $N_f \subseteq C$. First, we partition $\mathbb{N}$ into three sets R, A and B as in the proof of Proposition 1. Define $f : \mathbb{N} \to 2$ such that $\mathsf{dom}(f) := A \cup R$ and for all $i \in \mathsf{dom}(f)$, $f(i) = 1$. Through the inductive construction, every pair $(x, y) \in X \times X$ has been considered. Either (x, y) or (y, x) has been added to $A \cup R$. As a consequence, since every $z \in N_f$ takes value 1 for all pairs coded in $A \cup R$, it follows that every $z \in N_f$ codes a complete binary relation.

□

Table 1 below summarizes the results. None of the basic properties of binary relation is rare in doughnut topology whereas all of them are rare in the Cantor topology. Further, transitive or asymmetric or antisymmetric or irreflexive binary relations are not rare in Ellentuck topology whereas symmetric, or reflexive, or complete or linear binary relations are rare in Ellentuck topology. If we put together these three observations we can say that Ellentuck collections plays an important role as it allows to make a distinction between the former four properties (transitivity, asymmetry, antisymmetry and irreflexivity) on the one side, and the latter four properties (symmetry, reflexivity, completeness and linearity) on the other.

4 Egalitarian binary relations

In this section we consider anonymity- a procedural equity principle; strong equity - a consequentialist equity principle and the Pareto axiom- an efficiency principle. The utility space

Property	γ-small	ε-small	δ-small
Transitivity	yes	no	no
Asymmetry	yes	no	no
Antisymmetry	yes	no	no
Irreflexivity	yes	no	no
Symmetry	yes	yes	no
Reflexivity	yes	yes	no
Completeness	yes	yes	no
Linearity	yes	yes	no

Table 1: Basic properties of binary relations

X we consider here is countable. Take for instance $X := Y^N$, for $N \in \mathbb{N}$ and Y is any countable subset of $[0,1]$. One could choose $Y := \mathbb{Q} \cap [0,1]$ for example.

Proposition 3. *Let* $\mathbf{AN}, \mathbf{P}, \mathbf{SE} \subseteq 2^{\mathbb{N}}$ *be the coding set of all binary relations on* X *satisfying Anonimity, Pareto and Strong Equity, rispectively. Then* $\mathbf{AN}, \mathbf{P}$ *and* $\mathbf{SE}$ *consist of* γ*- and* ε*-small sets, but they are not* δ*-small.*

Proof. First we prove that there is $N_f \in \delta$ such that $N_f \subseteq \mathbf{AN}$. Let $\{e_n : n \in \mathbb{N}\}$ enumerate all streams in Y^N. Let $z \in 2^{\mathbb{N}}$ be defined as: for all $s, t \in Y^N$ $((s,t) \in \Re_z \Leftrightarrow s \sim_a t)$. Note that both $\{n \in \mathbb{N} : z(n) = 1\}$ and $\{n \in \mathbb{N} : z(n) = 0\}$ are infinite. Finally let $f : \mathbb{N} \to 2$ be the function such that $\mathsf{dom}(f) := \{n \in \mathbb{N} : z(n) = 1\}$ (in particular, $\forall n \in \mathsf{dom}(f)(f(n) = 1)$). Then note that every $x \in 2^{\mathbb{N}}$ is a code for an anonymous binary relation if it satisfies $x(n) = 1$ for every $n \in \mathsf{dom}(f)$. Hence, every $x \in N_f$ codes an anonymous binary relation, i.e., $N_f \subseteq \mathbf{AN}$.

Next we prove that $\mathbf{AN}$ is ε-small, and notice that a similar (and actually simpler argument) shows $\mathbf{AN}$ is γ-small as well. Fix arbitrarily an element $N_f \in \varepsilon$. Let $k \in \mathbb{N}$ large enough so that for all $n \geq k$, if $n \in \mathsf{dom}(f)$ then $f(n) = 0$. Now pick $m \geq k$ such that $z(m) = 1$. As in the proof of Proposition 2, we then distinguish two cases.

(1) If $m \in \mathsf{dom}(f)$, then $N_f \cap \mathbf{AN} = \emptyset$.

(2) If $m \notin \mathsf{dom}(f)$, then define the partial function $g : \mathbb{N} \to 2$ with $\mathsf{dom}(g) := \mathsf{dom}(f) \cup \{m\}$ as:

$$g(i) := \begin{cases} f(i) & \text{if } i \in \mathsf{dom}(f) \\ 0 & \text{if } i = m. \end{cases}$$

Note that $N_g \in \varepsilon$ and $N_g \subseteq N_f$. Moreover, since $g(m) = 0$, it follows that for any binary relation coded by any $z \in N_g$ we can find a pair (t, t') such that $t \sim_a t'$ but (t, t') is not in the binary relation coded by z; which gives $N_g \cap \mathbf{AN} = \emptyset$.

Similar argument holds for Paretian binary relation. First, we prove that there exists $N_f \in \delta$ such that $N_f \subseteq \mathbf{PA}$. Define $A(n) \subset \mathbb{N}$ and $B(n) \subset (\mathbb{N} \setminus R)$ (where R is as defined in 1) recursively as follows:

- Start from $q_1 = (x_{j_1}, x_{m_1})$. Let $k(1) \in \mathbb{N}$ be such that $q_{k(1)} = (x_{m_1}, x_{j_1})$. If (x_{j_1}, x_{m_1}) satisfies the Pareto condition, then put $A(1) := \{1\}$ and $B(1) := \{k(1)\}$. Otherwise let $A(1) = \emptyset$ and $B(1) = \emptyset$.

- Assume $A(n-1)$ and $B(n-1)$ have been defined for $n \geq 2$ and pick $q_n = (x_{j_n}, x_{m_n})$; if (x_{j_n}, x_{m_n}) satisfies the Pareto condition, then put $A(n) := A(n-1) \cup \{n\}$ and $B(n) := B(n-1) \cup \{k(n)\}$; otherwise let $A(n) = A(n-1)$ and $B(n) := B(n-1)$.

Finally put $A := \bigcup_{n \in \mathbb{N}} A(n)$ and $B := \bigcup_{n \in \mathbb{N}} B(n)$. By construction, both A and B are infinite. Set A enumerates all pairs of alternatives (x_i, x_l) such that $x_i <_p x_l$. For each element in set A, set B enumerates all the corresponding opposite pairs of alternatives (x_l, x_i) such that $x_l <_p x_i$. Since R is infinite, the complement of $A \cup B$ is also infinite. Then define the partial function $f : \mathbb{N} \to 2$ with $\mathsf{dom}(f) := A \cup B$ as:

$$f(n) := \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \in B. \end{cases}$$

We claim that every $z \in N_f$ codes a Paretian binary relation. In fact, if $n \in N$ is such that the pair (x_{j_n}, x_{m_n}) satisfies the Pareto axiom, then $n \in A$ and the related $k(n) \in B$ and therefore $z(n) = f(n) = 1$ (which means $(x_{j_n}, x_{m_n}) \in \Re_z$) and $z(k(n)) = f(k(n)) = 0$ (which means $(x_{m_n}, x_{j_n}) \notin \Re_z$).

The proof to show that **PA** is ε-small follows the same line as for **AN**.

We leave to the reader the similar details for **SE** as well. □

Table 2 summarizes these results.

Property	γ-small	ε-small	δ-small
Anonymity	yes	yes	no
Paretian	yes	yes	no
Strong equity	yes	yes	no

Table 2: Equity and efficiency properties of preference relations

5 Concluding remarks

In this paper, we have used the Ellentuck and doughnut topologies (from the branch of descriptive set theory in the mathematical logic literature) to investigate the rarity of binary relations endowed with useful basic features (transitive, asymmetric, etc.) and economic features (Paretian, anonymous, equity). Propositions 1 and 2 show that these binary relations are not rare in the finest (doughnut) topology. The Ellentuck topology yields mixed results. Transitive, asymmetric or antisymmetric binary relations are not rare whereas complete, reflexive or symmetric binary relations are rare in Ellentuck topology. These results lead to a better understanding of the distinct nature of Cantor topology compared to the Ellentuck and doughnut topologies. Finally, Propositions 3 on the equitable or Paretian binary relations show that none of them are rare in doughnut topology.

In future research, we intend to study the pervasiveness or rarity of the binary relations endowed with desirable features (basic properties, equity or efficiency axioms) on the set of alternatives X containing uncountably many elements using analytical tools from the generalized descriptive set theory.

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