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A core-equilibrium convergence in a public goods economy

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Abstract

This paper deals with a core-equilibrium equivalence in an economy with public goods where preferences of consumers display warm glow effects. We demonstrate that provided that each consumer becomes satiated to other consumers' provision, it holds that for a sufficiently large economy the core shrinks and the set of Edgeworth allocations is nonempty. Moreover, we show that an Edgeworth allocation could be decentralized as a warm glow equilibrium.

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1 Introduction

The seminal contribution of Debreu and Scarf (1963) shows that, in a pure exchange economy, the core allocations shrink to the Walrasian equilibrium as the set of consumers is appropriately enlarged. The core consists of allocations of resources such that no coalition could achieve a preferred outcome for its members by seceding from the grand coalition and proposing another allocation that is reachable within its own resources. The Debreu and Scarf contribution, which is a lucid formulation of an earlier conjecture by Edgeworth (1881), has gained its prominence as a justification of the emergence of competitive behavior as a result of social stability.

In economies with public goods, various attempts have been made to investigate whether a similar core-equilibrium convergence holds for the Lindahl price-based mechanism. This is of paramount importance, as it would provide the Lindahl equilibrium with the same solid foundation of competitive equilibrium. Unfortunately, it turns out that this type of convergence is the exception rather than the rule. The literature is furnished with either robust examples of non-convergence (for example, see Muench (1972), Milleron (1972), Champsaur, Roberts and Rosenthal (1975) and Buchholz and Peters (2007)) or few context specific convergences. This is hardly surprising since intuitively, unlike the case of a pure exchange economy, potential blocking coalitions are more likely to fall short of the resources available to the grand coalition to produce public goods. Hence unless the benefit of an ever-increasing public goods provision is limited when the economy grows large, the core equilibrium convergence is deemed to fail. The work of Wooders (1983) and Conley (1994) provides valuable insights into economies with public goods where the above limitation may hold. The game theoretic approach of Wooders (1983) imposes the assumption of per-capita boundedness on the equal treatment payoffs of replica games to ensure the nonemptiness of an approximate limit core. Conley (1994) evokes the possibility of asymptotic satiation in public goods consumption due to the resulting colossal magnitude of the aggregate supply of public goods in large replica economies.

More recently, Vasil′ev, Weber and Wiesmeth (1995) and Florenzano and del Mercato (2006) show a subtle convergence of non-standard core allocations to Lindahl equilibrium. Given that the set of core allocations is usually bigger than the Lindahl allocations, Vasil′ev, Weber and Wiesmeth (1995) and Florenzano and del Mercato (2006) bypass this difficulty by constructing a sequence of artificial replica economies where the public goods provision of each coalition is normalized by its size. More generally, as in the clubs/local public goods literature, where the public goods are subject to crowding and congestion, the social stability underlying the formation of communities and groups providing these goods will eventually settle the economy in a competitive equilibrium (for example, see Wooders (1989, 1997) and Conley and Smith (2005) for a survey).

The existence of Lindahl equilibrium in economies with public goods was first formalized by Foley (1967, 1970) (see also Fabre-Sender (1969), Milleron (1972), Roberts (1974), Bergstrom (1976) and Myles (1995)). Foely's approach consists of embedding the public goods economy into a larger private goods economy wherein each consumer is the only buyer of his own copy of each public good. The existence of Lindahl equilibrium is then established by resorting to standard existence results for private goods economy. In the public goods literature, lately, the warm glow model, where consumers receive a direct benefit from their own public goods provision, has been put forward by Andreoni (1989, 1990) (see also Becker (1974), Cornes and Sandler (1984)) as an alternative description of public goods provision. In a recent paper, Allouch (2009) introduces a Lindahl-like competitive equilibrium for a warm glow economy and provides the three fundamental theorems of general equilibrium (the two Welfare Theorems and existence). It is worth noting that the warm glow equilibrium coincides with the Lindahl equilibrium if we consider a standard formalization of utility functions. A natural question then arises: "*Under what circumstances do the core allocations converge to the warm glow equilibrium?*"

Fortunately, one possible answer to the above question comes from the literature on warm glow itself:

> "For example, as the size of the population increases, choosing a contribution level becomes more and more like picking the level of consumption for any conventional good. In the limit, the contributor simply weighs the relative merits of spending money on two different private goods, x_i and g_i; the effect on his well-being through G becomes negligible."(p. 58, Bernheim and Rangel (2005) "Behavioral public economics: welfare and policy analysis with non-standard decision makers" *NBER Working Paper,* 11518.)

And,

"Another way to see this intuitively is that, as the size of the charity grows, all giving due to altruism will be crowded out, leaving only giving due to warm-glow. This accords naturally with the observation that giving 100 dollars to an organization that collects millions is motivated more by an admiration for the organization than for any measurable effect of the marginal donation."(p. 1223, Andreoni (2006) "Philanthropy,"in *the Handbook of Giving, Reciprocity and Altruism.*)

In this paper we attempt to articulate the above observations into an economic assumption in order to establish that core allocations converge to the warm glow equilibrium. We emphasize that each consumer's public goods provision is more driven by the act of giving itself rather than the aggregate provision of public goods. Thus eventually an increase in public goods provision by other consumers has no effect on the consumer's welfare. Formally, our main economic assumption stipulates that, beyond a threshold of other consumers' public goods provision, each consumer benefits only from his own public goods provision and private goods consumption.

In the following, Section 2 is devoted to the model, Section 3 defines core and Edgeworth allocations and shows the non-vacuity of the set of Edgeworth allocations. In Section 4 we introduce the concept of warm glow equilibrium and state our core-equilibrium equivalence result. Section 5 is an Appendix containing proofs.

2 The model

We consider a public goods economy $\mathcal{E}$ with $i = 1, \ldots, N$ consumers, $l = 1, \ldots, L$ private goods and $k = 1, \ldots, K$ public goods. A consumption bundle of private goods is denoted by $x = (x^1, \ldots, x^L) \in \mathbb{R}^L_+$ and a consumption bundle of public goods is denoted by $g = (g^1, \ldots, g^K) \in \mathbb{R}^K_+$. The private and public goods consumption set is $\mathbb{R}^{L+K}_+$, for each consumer i. The production technology for public goods is described by an aggregate production set $Y \subset \mathbb{R}^{L+K}$. A typical production plan will be written (y, g), where $y \in \mathbb{R}^L$ denotes *inputs of private goods* and $g \in \mathbb{R}^K$ denotes *outputs of public goods.* Each consumer i has an endowment of private goods, denoted by $w_i \in \mathbb{R}^L_{++}$, and has no endowment of public goods. The preferences of each consumer i may be represented by a utility function $u_i(x_i, g_i, G_{-i})$, where x_i is consumer's i private goods consumption, g_i is consumer's i public goods consumption, and $G_{-i} = \sum_{j \neq i} g_j$ is the total provision of public goods minus consumer's i provision. The utility function u_i satisfies the following properties:

[**A.1**] **Monotonicity**: The utility function $u_i(\cdot, \cdot, \cdot)$ is increasing. Moreover, given any $G_{-i} \in \mathbb{R}^K_+$, the function $u_i(\cdot, \cdot, G_{-i})$ is strictly increasing on $\mathbb{R}^L_+ \times \mathbb{R}^K_{++}$.

[**A.2**] **Continuity**: The utility function $u_i(\cdot, \cdot, \cdot)$ is continuous.

[**A.3**] **Convexity**: The utility function $u_i(\cdot, \cdot, \cdot)$ is quasi-concave.

[**A.4**] **Warm glow indispensability**: For every $(x_i, g_i, G_{-i}) \in \mathbb{R}^{L+K}_+$, if $g_i \notin \mathbb{R}^K_{++}$ then $u_i(x_i, g_i, G_{-i}) = \inf u_i(\cdot, \cdot, \cdot)$.

For simplicity, we consider a constant returns to scale technology. Thus we assume that Y is a closed convex cone with vertex the origin; satisfying the usual conditions of irreversibility, no free production, and free disposal. In addition, we assume the possibility of producing public goods, that is, $Y \cap (\mathbb{R}^L \times \mathbb{R}^K_{++}) \neq \emptyset$.

3 Core and Edgeworth allocations

Let S be a nonempty subset of N. An allocation $((x_i, g_i), i \in S) \in \mathbb{R}^{(L+K)|S|}_+$ is S-feasible if

$$(\sum_{i \in S}(x_i - w_i), \sum_{i \in S} g_i) \in Y.$$

For simplicity of notations, an N-feasible allocation will simply be called a feasible allocation.

A coalition $S \subset N$ can *improve upon* an allocation $((\overline{x}_i, \overline{g}_i), i \in N)$ if there exists an S-feasible allocation $((x_i, g_i), i \in S)$, such that

$$u_i(x_i, g_i, \sum_{j \in S \setminus \{i\}} g_j) > u_i(\overline{x}_i, \overline{g}_i, \overline{G}_{-i}), \text{ for each consumer } i \in S.$$

That is to say, coalition S could do better for its members by breaking away from the grand coalition and proposing another allocation that is achievable with its own resources. An allocation $((\overline{x}_i, \overline{g}_i), i \in N)$ is in the *core* if it is feasible and cannot be improved upon by any coalition $S \subset N$.

For each positive integer r, we define the r^{th} *replica economy*, denoted by $\mathcal{E}_r$, as the economy with a set of consumers

$$N_r = \{(i,q) \mid i = 1,\dots,N \text{ and } q = 1,\dots,r\}.$$

Consumer (i,q) is called the q^{th} *consumer of type* i. It will be the case that all consumers of type i are identical in terms of their consumption sets, endowments and preferences to consumer i. Let S be a nonempty subset of N_r. An allocation $((x_{(i,q)}, g_{(i,q)}), (i,q) \in S)$ is S-feasible in the economy $\mathcal{E}_r$ if

$$\Big(\sum_{(i,q)\in S} (x_{(i,q)} - w_{(i,q)}), \sum_{(i,q)\in S} g_{(i,q)}\Big) \in Y.$$

For each integer r, we define the r^{th} replica of allocation $((x_i, g_i), i \in N)$, denoted by $((x_{(i,q)}, g_{(i,q)}), (i,q) \in N_r)$, as follows:

$$x_{(i,q)} = x_i \text{ and } g_{(i,q)} = g_i, \text{ for each } (i,q) \in N_r.$$

That is, in the r^{th} replica economy $\mathcal{E}_r$, each of the r^{th} replica consumers of type i has the same private goods and public goods consumption as consumer i. An allocation $((\overline{x}_i, \overline{g}_i), i \in N)$ is called an *equal treatment core allocation* of the r^{th} replica economy $\mathcal{E}_r$ if the r^{th} replica of allocation $((\overline{x}_i, \overline{g}_i), i \in N)$ is in the core of $\mathcal{E}_r$. The set of all equal treatment core allocations of $\mathcal{E}_r$ is called the *equal treatment core* and is denoted by $\mathcal{C}^r$.

Finally, an allocation $((\overline{x}_i, \overline{g}_i), i \in N)$ is an *Edgeworth allocation*, if for each positive integer r, $((\overline{x}_i, \overline{g}_i), i \in N)$ is in the equal treatment core of the r^{th} replica economy $\mathcal{E}_r$, that is,

$$((\overline{x}_i, \overline{g}_i), i \in N) \in \bigcap_{r=1}^{\infty} \mathcal{C}^r.$$

3.1 Non-vacuity of the set of Edgeworth allocations

Brenheim and Rangel (2005) and Andreoni (2006) argue that asymptotically consumers' charitable giving is due more to the act of giving itself rather than concerns about the aggregate provision of public goods. Our main economic assumption attempts to formalize the above argument.

[WGD] Warm glow dominance[1] For every consumer $i \in N$, there exists a bundle of public goods $G^*_{-i} \in \mathbb{R}^K_{++}$, such that for all $(x_i, g_i, G_{-i}) \in \mathbb{R}^{L+2K}_{+}$ with $G_{-i} \geq G^*_{-i}$, it holds that

$$u_i(x_i, g_i, G^*_{-i}) = u_i(x_i, g_i, G_{-i}).$$

[1]We borrowed this term from Andreoni (2006).

The [WGD] assumption ensures that beyond a public goods bundle[2] of other consumers' provision, each consumer benefits only from his public goods provision and his private goods consumption. It is worth noting that the [WGD] assumption does not imply the asymptotic satiation assumption in public goods of Conley (1994) since the utility functions could still be unbounded due to an ever-increasing own provision of public goods.

In public goods economies, it is well known that the core does not shrink and may well expand, unless the returns to coalition size are limited. Our theorem below shows that, under the warm glow dominance assumption, for a sufficiently large replica economy, the core shrinks and the set of Edgeworth allocations is nonempty.

Theorem 1. Assume [A.1]-[A.4] and [WGD]. Then there exists a positive integer r^* such that for each $r \geq r^*$, it holds that

$$\mathcal{C}^{r+1} \subset \mathcal{C}^r \text{ and } \bigcap_{r=r^*}^{\infty} \mathcal{C}^r \neq \emptyset.$$

Proof of Theorem 1. We first construct an auxiliary private goods production economy $\hat{\mathcal{E}}$ with N consumers. Each consumer i is described by a consumption set $\mathbb{R}_+^{L+K}$, an endowment $(w_i, 0) \in \mathbb{R}_+^{L+K}$, and a utility function $\hat{u}_i$ defined as follows

$$\hat{u}_i(x_i, g_i) = u_i(x_i, g_i, G_{-i}^*), \quad \text{for all } (x_i, g_i) \in \mathbb{R}_+^{L+K}.$$

The production technology of the auxiliary economy $\hat{\mathcal{E}}$ is characterized by the production set Y. Hence it is obvious that the set of feasible allocations of the auxiliary economy $\hat{\mathcal{E}}$ coincides with the set of feasible allocations of the economy $\mathcal{E}$. For each positive integer r, let us consider $\hat{\mathcal{C}}^r$ the set of equal treatment allocations in the core of the r^{th} replica of the economy $\hat{\mathcal{E}}$. From standard results on the non-vacuity of the core and the set of Edgeworth allocations for a private goods production economy (for example, see Aliprantis,

[2]This public goods bundle may not be feasible in the economy.

Brown and Burkinshaw (1990), Florenzano (1990, 2003)) it holds that for each positive integer r,

$$\hat{\mathcal{C}}^{r+1} \subset \hat{\mathcal{C}}^{r} \text{ and } \bigcap_{r=1}^{\infty} \hat{\mathcal{C}}^{r} \neq \emptyset.$$

We claim that there exists a positive integer r^* such that, for each $r \geq r^*$, it holds that

$$\mathcal{C}^{r} = \hat{\mathcal{C}}^{r}. \tag{1}$$

We start with the following observation: from the definition of the auxiliary utility function $\hat{u}_i$, it is easy to see that for any allocation $((\overline{x}_i, \overline{g}_i), i \in N)$, whenever $(r-1)\overline{g}_i \geq G^*_{-i}$, it holds that

$$((\overline{x}_i, \overline{g}_i), i \in N) \in \mathcal{C}^{r} \text{ if and only if } ((\overline{x}_i, \overline{g}_i), i \in N) \in \hat{\mathcal{C}}^{r}.$$

Suppose (1) were not true. Then, without loss of generality, one could show that for each positive integer r, there exists $n_r \geq r$ and an allocation $((\overline{x}_i^{n_r}, \overline{g}_i^{n_r}), i \in N) \in \hat{\mathcal{C}}^{n_r}$, such that for some $i_0 \in N$, it holds that

$$(n_r - 1)\overline{g}_{i_0}^{n_r} \ngeq G^*_{-i_0}. \tag{2}$$

By compactness of the feasible set for the auxiliary economy $\hat{\mathcal{E}}$ and without loss of generality one could assume that $((\overline{x}_i^{n_r}, \overline{g}_i^{n_r}), i \in N)$ converges to $((\overline{x}_i^*, \overline{g}_i^*), i \in N)$. It then follows from (2) that $\overline{g}_i^* \notin \mathbb{R}^K_{++}$. Then by the warm glow indispensability assumption and the possibility of producing public goods, it follows that for sufficiently large n_r, the allocation $((\overline{x}_i^{n_r}, \overline{g}_i^{n_r}), i \in N)$ could be blocked by any consumer of type i_0, in the n_r^{th} replica of the economy $\hat{\mathcal{E}}$. This contradicts the fact that $((\overline{x}_i^{n_r}, \overline{g}_i^{n_r}), i \in N) \in \hat{\mathcal{C}}^{n_r}$.□

4 Warm glow equilibrium

In a recent paper, Allouch (2009) introduces the warm glow equilibrium concept as a competitive equilibrium for a warm glow economy. Similar to the Lindahl equilibrium, the warm glow equilibrium[3] provides a decentralized

[3]It is worth noting that the warm glow equilibrium coincides with the Lindahl equilibrium if we consider the standard public good model, where preferences of consumer i are represented by the utility function $u_i(x_i, g_i + G_{-i})$ instead of $u_i(x_i, g_i, G_{-i})$.

price mechanism achieving efficient outcomes. In a warm glow equilibrium each consumer faces a common price for his private goods consumption, a personalized price for his own public goods provision, and another personalized price for other consumers' public goods provision. These personalized prices arise from the externalities brought about by each consumer public goods provision.

Definition: A warm glow equilibrium is $((\overline{x}_i, \overline{g}_i, \pi_i, \pi_{-i})_{i\in N}, p, p^g)$, where $((\overline{x}_i, \overline{g}_i), i \in N)$ is a feasible allocation, $p \in \mathbb{R}^L_+$ is a price system for private goods, $p^g \in \mathbb{R}^K_+$ a price system for public goods, $\pi_i \in \mathbb{R}^K_+$ is the personalized price of consumer's i own public goods provision, and $\pi_{-i} \in \mathbb{R}^K_+$, is consumer's i personalized price for other consumers' public goods provision, such that

(i). For all $(y, g) \in Y$,

$$(p, p^g) \cdot (y, g) \leq (p, p^g) \cdot (\sum_i (\overline{x}_i - w_i), \sum_i \overline{g}_i) = 0;$$

(ii). for each consumer $i \in N$,

$$p \cdot x_i + \pi_i \cdot g_i + \pi_{-i} \cdot G_{-i} = p \cdot w_i,$$

and if

$$u_i(x_i, g_i, G_{-i}) > u_i(\overline{x}_i, \overline{g}_i, \overline{G}_{-i})$$

then

$$p \cdot x_i + \pi_i \cdot g_i + \pi_{-i} \cdot G_{-i} > p \cdot w_i;$$

(iii). for each consumer $i \in N$,

$$\pi_i + \sum_{j\neq i} \pi_{-j} = p^g.$$

Condition (i) is the profit maximization, Condition (ii) is the utility maximization and Condition (iii) ensures the personalized prices for each consumer public goods provision sum up to the public goods price.

We now show that the warm glow equilibrium belongs to the core of the economy. It is worth noting that, similar to other competitive equilibrium concepts, our proof could be easily extended to show that the warm glow equilibrium belongs to the core of each replica economy.

Theorem 2. If $((\overline{x}_i, \overline{g}_i, \pi_i, \pi_{-i})_{i\in N}, p, p^g)$ is a warm glow equilibrium, then $((\overline{x}_i, \overline{g}_i), i \in N)$ is in the core.

Proof of Theorem 2. First, it follows from the definition of warm glow equilibrium that $((\overline{x}_i, \overline{g}_i), i \in N)$ is feasible. Now suppose that there is a coalition $S \subset N$ and an S-feasible allocation $((x_i, g_i), i \in S)$ such that

$$u_i(x_i, g_i, \sum_{j\in S\setminus\{i\}} g_j) > u_i(\overline{x}_i, \overline{g}_i, \overline{G}_{-i}), \quad \text{for each } i \in S.$$

From (ii) in the definition of warm glow equilibrium, for each consumer i, it holds that

$$p \cdot x_i + \pi_i \cdot g_i + \pi_{-i} \cdot \sum_{j\in S\setminus\{i\}} g_j > p \cdot w_i.$$

Summing up the above inequalities over $i \in S$, it holds that

$$\sum_{i\in S} p \cdot x_i + \sum_{i\in S}(\pi_i + \sum_{j\in S\setminus\{i\}} \pi_{-j}) \cdot g_i > \sum_{i\in S} p \cdot w_i. \tag{3}$$

We may now extend the allocation $((x_i, g_i), i \in S)$ in the following way. For each $i \notin S$, we set $(x_i, g_i) = (w_i, 0)$. Since $((x_i, g_i), i \in S)$ is an S-feasible allocation it follows that $((x_i, g_i), i \in N)$ is a feasible allocation.
Then (3) implies that

$$\sum_{i\in N} p \cdot x_i + \sum_{i\in N}(\pi_i + \sum_{j\in N\setminus\{i\}} \pi_{-j}) \cdot g_i > \sum_{i\in N} p \cdot w_i.$$

Since $p^g = \pi_i + \sum_{j\in N\setminus\{i\}} \pi_{-i}$, it follows that

$$p \cdot \sum_{i\in N}(x_i - w_i) + p^g \cdot \sum_{i\in N} g_i > 0,$$

which contradicts (i) in the definition of equilibrium .□

We now show that an Edgeworth allocation could be decentralized as a warm glow equilibrium. In public goods economies, the fundamental insight is to construct an auxiliary economy in which the public goods consumption set is expanded so that each consumer is the only buyer of his own copy of the public goods bundle. Thus, unlike the standard private goods economy, each consumer in the auxiliary economy faces a personalized price for the aggregate public goods provision. In the following, we construct an auxiliary economy in the spirit of Foley (1970). However, our decentralization argument differs from Foley's argument since we expand the economy in a specific way in order to take account of warm glow effects in preferences. Moreover, we consider an individual preferred set for each consumer rather than an aggregate preferred set for the economy.

Theorem 3. Assume [A.1]-[A.4]. Let $((\overline{x}_i, \overline{g}_i), i \in N)$ be an Edgeworth allocation of the economy satisfying, $\overline{g}_i \in \mathbb{R}^K_{++}$, for every $i \in N$. Then there is a price system $((\pi_i, \pi_{-i})_{i\in N}, p, p^g) \neq 0$ such that $((\overline{x}_i, \overline{g}_i, \pi_i, \pi_{-i})_{i\in N}, p, p^g)$ is a warm glow equilibrium.

Proof of Theorem 3. See the Appendix.

The following Theorem concludes our results.

Theorem 4. Assume [A.1]-[A.4] and [WGD]. Then there exists a positive integer r^*, such that the r^* replica economy $\mathcal{E}_{r^*}$ has a warm glow equilibrium. Moreover, the set of warm glow equilibria of the economy $\mathcal{E}_{r^*}$ is equivalent to the set of Edgeworth allocations.

Proof of Theorem 4. This is immediate from Theorem 1, Theorem 2 and Theorem 3. Indeed, Theorem 1 states that eventually the core shrinks and the set of Edgeworth allocations is nonempty. Theorem 2 states that a warm glow equilibrium is in the core and Theorem 3 shows that an Edgeworth allocation could be decentralized as a warm glow equilibrium.

5 Appendix

Proof of Theorem 3. Let $((\overline{x}_i, \overline{g}_i), i \in N)$ be an Edgeworth allocation for the economy satisfying, $\overline{g}_i \in \mathbb{R}^K_{++}$, for every $i \in N$. First, for each consumer i, we define the set $\Gamma_i \subset \mathbb{R}^{L+2NK}$; where N is the number of consumers, L is the number of private goods and K is the number of public goods, as follows:

$$\Gamma_i = \{(x_i - w_i, 0, 0, \ldots, g_i, G_{-i}, \ldots, 0, 0) \mid u_i(x_i, g_i, G_{-i}) > u_i(\overline{x}_i, \overline{g}_i, \overline{G}_{-i})\}.$$

The set Γ_i is consumer i's expanded preferred set, listing his net trade in private goods $x_i - w_i$ and each consumer $j \in N$ public goods provision and complementary public goods provision (g_j, G_{-j}) such that $(g_j, G_{-j}) = (0, 0)$ for all $j \neq i$ and (x_i, g_i, G_{-i}) is strictly preferred to $(\overline{x}_i, \overline{g}_i, \overline{G}_{-i})$ by consumer i.

Since [A.1], the set Γ_i is nonempty for each consumer i. In addition, it is easy to check that the convexity of the preferences implies that Γ_i is convex for each consumer i. Let Γ denote the convex hull of the union of the sets Γ_i, $i = 1, \ldots, N$. It is worth noting that the convex hull of the union of a finite number of convex sets may be written as the convex combination of these sets.

Now we define the set:

$$\widetilde{Y} = \{(y, g_1, G_{-1}, \ldots, g_N, G_{-N}) \mid \text{ for each } i,\ G_{-i} = \sum_{j \neq i} g_j \text{ and } (y, \sum_{j \in N} g_j) \in Y\}.$$

The set $\widetilde{Y}$ is a convex cone with vertex the origin since Y is a convex cone with vertex the origin. We claim that

$$\Gamma \cap \widetilde{Y} = \emptyset.$$

To see this, assume, on the contrary, that $\Gamma \cap \widetilde{Y} \neq \emptyset$. Then there exists $(x_i - w_i, 0, 0, \ldots, g_i, G_{-i}, \ldots, 0, 0) \in \Gamma_i$ and $\lambda \in \mathbb{R}^{\mathbb{N}}_+$ such that $\sum_i \lambda_i = 1$, and

$$\sum \lambda_i (x_i - w_i, 0, 0, \ldots, g_i, G_{-i}, \ldots, 0, 0) \in \widetilde{Y}.$$

Let $S = \{i \mid \lambda_i > 0\}$. It is obvious that $S \neq \emptyset$ since $\sum_i \lambda_i = 1$. For each $i \in S$ and each positive integer n, let n_i be the smallest integer which is greater than or equal to $n\lambda_i$. For each $i \in S$, define

$$(x_i^n, g_i^n, G_{-i}^n) = \frac{n\lambda_i}{n_i}(x_i, g_i, G_{-i}) + (1 - \frac{n\lambda_i}{n_i})(w_i, 0, 0). \tag{4}$$

From continuity of preferences, for all n sufficiently large, it holds that

$$u_i(x_i^n, g_i^n, G_{-i}^n) > u_i(\overline{x}_i, \overline{g}_i, \overline{G}_{-i}), \quad \text{for each } \ i \in S.$$

It follows from (4) that

$$\sum_{i\in S}(\frac{n_i}{n})\frac{n\lambda_i}{n_i}(x_i - w_i, 0, 0, \ldots, g_i, G_{-i}, \ldots, 0, 0) \in \widetilde{Y}.$$

Then

$$\sum_{i\in S}\frac{n_i}{n}(x_i^n - w_i, 0, 0, \ldots, g_i^n, G_{-i}^n, \ldots, 0, 0) \in \widetilde{Y}.$$

Since $\widetilde{Y}$ is a cone with vertex zero, it holds that

$$\sum_{i\in S} n_i(x_i^n - w_i, 0, 0, \ldots g_i^n, G_{-i}^n, \ldots, 0, 0) \in \widetilde{Y}.$$

Thus we have constructed a blocking coalition, which is a contradiction to the assumption that $((\overline{x}_i, \overline{g}_i), i \in N)$ is an Edgeworth allocation. Therefore $\Gamma \cap \widetilde{Y} = \emptyset$.

From Minkowski's separating hyperplane theorem, there exists a hyperplane with normal $(p, \pi_1, \pi_{-1}, \ldots, \pi_N, \pi_{-N}) \neq 0$, and a scalar r such that

(i). for all $(y, g_1, G_{-1}, \ldots, g_N, G_{-N}) \in \Gamma$

$$(p, \pi_1, \pi_{-1}, \ldots, \pi_N, \pi_{-N}) \cdot (y, g_1, G_{-1}, \ldots, g_N, G_{-N}) \geq r,$$

(ii). for all $(y, g_1, G_{-1}, \ldots, g_N, G_{-N}) \in \widetilde{Y}$

$$(p, \pi_1, \pi_{-1}, \ldots, \pi_N, \pi_{-N}) \cdot (y, g_1, G_{-1}, \ldots, g_N, G_{-N}) \leq r.$$

Since $\widetilde{Y}$ is a closed convex cone with vertex zero, we can choose $r = 0$. It follows from (i) in the separation theorem that for any consumer i, and any consumption bundle (x_i, g_i, G_{-i}) such that $u_i(x_i, g_i, G_{-i}) > u_i(\overline{x}_i, \overline{g}_i, \overline{G}_{-i})$, it holds that

$$p \cdot (x_i - w_i) + \pi_i \cdot g_i + \pi_{-i} \cdot G_{-i} \geq 0. \tag{5}$$

Thus by continuity and monotonicity of preferences with respect to private goods, we obtain

$$p \cdot (\overline{x}_i - w_i) + \pi_i \cdot \overline{g}_i + \pi_{-i} \cdot \overline{G}_{-i} \geq 0.$$

Summing up the above inequalities over $i \in N$ and rearranging terms, we get

$$(p, \pi_1, \pi_{-1}, \ldots, \pi_N, \pi_{-N}) \cdot (\sum(\overline{x}_i - w_i), \overline{g}_1, \overline{G}_{-1}, \ldots, \overline{g}_N, \overline{G}_{-N}) \geq 0. \quad (6)$$

By feasibility of $((\overline{x}_i, \overline{g}_i), i \in N)$ and (ii) in the separation theorem, it holds that

$$(p, \pi_1, \pi_{-1}, \ldots, \pi_N, \pi_{-N}) \cdot (\sum(\overline{x}_i - w_i), \overline{g}_1, \overline{G}_{-1}, \ldots, \overline{g}_N, \overline{G}_{-N}) \leq 0. \quad (7)$$

Hence it follows from (6) and (7) that

$$(p, \pi_1, \pi_{-1}, \ldots, \pi_N, \pi_{-N}) \cdot (\sum(\overline{x}_i - w_i), \overline{g}_1, \overline{G}_{-1}, \ldots, \overline{g}_N, \overline{G}_{-N}) = 0. \quad (8)$$

And, therefore

$$p \cdot (\overline{x}_i - w_i) + \pi_i \cdot \overline{g}_i + \pi_{-i} \cdot \overline{G}_{-i} = 0. \quad (9)$$

We claim that, for any two consumers j_1 and j_2, it holds that

$$\pi_{j_1} + \sum_{i \neq j_1} \pi_{-i} = \pi_{j_2} + \sum_{i \neq j_2} \pi_{-i}.$$

Suppose this were not the case, then, without loss of generality, one could assume that for some public good, say the k^{th}, it holds that

$$\pi^k_{j_1} + \sum_{i \neq j_1} \pi^k_{-i} > \pi^k_{j_2} + \sum_{i \neq j_2} \pi^k_{-i}.$$

Let δ_k be a vector in $\mathbb{R}^K_+$ consisting of one unit of the k^{th} public good and nothing else. For a small enough $\varepsilon > 0$, let us consider the following public goods bundle, $\overline{G}^\varepsilon = (\overline{g}^\varepsilon_i, \ldots, \overline{g}^\varepsilon_N)$, defined as follows

$$\overline{G}^\varepsilon = \begin{cases} \overline{g}^\varepsilon_{j_1} = \overline{g}_{j_1} + \varepsilon\delta_k, & \\ \overline{g}^\varepsilon_{j_2} = \overline{g}_{j_2} - \varepsilon\delta_k, & \\ \overline{g}^\varepsilon_i = \overline{g}_i, & \text{if } i \in N \setminus \{j_1, j_2\}. \end{cases}$$

It is obvious that

$$(\sum(\overline{x}_i - e_i), \overline{g}^\varepsilon_1, \overline{G}^\varepsilon_{-1}, \ldots, \overline{g}^\varepsilon_N, \overline{G}^\varepsilon_{-N}) \in \widetilde{Y}.$$

Moreover, since (8) it follows that

$$(p, \pi_1, \pi_{-1}, \ldots, \pi_N, \pi_{-N}) \cdot (\sum(\overline{x}_i - e_i), \overline{g}_1^{\varepsilon}, \overline{G}_{-1}^{\varepsilon}, \ldots, \overline{g}_N^{\varepsilon}, \overline{G}_{-N}^{\varepsilon}) > 0,$$

but, this contradicts property (i) of the separation theorem. Thus we set up

$$p^g = \pi_i + \sum_{j \neq i} \pi_{-j}, \text{ for all } i \in N.$$

In view of this, property (ii) of the separation theorem and (8) imply that for all $(y, g) \in Y$,

$$(p, p^g) \cdot (y, g) \leq (p, p^g) \cdot (\sum_i (\overline{x}_i - w_i), \sum_i \overline{g}_i) = 0.$$

This proves (i) and (iii) in the definition of warm glow equilibrium.

From [A.1] and the separation theorem it follows that $p \in \mathbb{R}_+^L \setminus \{0\}$ and for each consumer i, $(\pi_i, \pi_{-i}) \in \mathbb{R}_+^K \setminus \{0\} \times \mathbb{R}_+^K$. We now show that for any consumer i and any consumption bundle (x_i, g_i, G_{-i}), such that $u_i(x_i, g_i, G_{-i}) > u_i(\overline{x}_i, \overline{g}_i, \overline{G}_{-i})$, it holds that

$$p \cdot x_i + \pi_i \cdot g_i + \pi_{-i} \cdot G_{-i} > p \cdot w_i.$$

Assume that this were not the case. By indispensability of warm glow provision, it follows that $g_i \in \mathbb{R}_{++}^K$. Then there exists $g_i' \in \mathbb{R}_+^K$, such that $g_i' << g_i$. Therefore, by quasi-concavity and continuity, along the line joining (x_i, g_i', G_{-i}) and (x_i, g_i, G_{-i}), there is a point in the consumption set of i which is strictly preferred to $(\overline{x}_i, \overline{g}_i, \overline{G}_{-i})$ and costs strictly less than $p \cdot w_i$. This contradicts (5). This and (9) prove (ii) in the definition of warm glow equilibrium.□

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