Economics Bulletin

Appendix and Supplemental material not intended for publication-Round 2

Submission Number:EB-13-00235

Mathematical Appendix

The authors wish to thank an anonymous referee, Wouter Den Haan, Etienne Lehmann, Shouyoung Shi, and Randy Wright for useful advice. They are also thankful to the department of economics at UQÀM and the CIRPÉE for their hospitality. Financial support from Junta de Andalucía through grants SEJ-552 and SEJ-1645 and from the Spanish Ministry of Science and Innovation through grants ECO 2008-03674/ECON and ECO2011-29355 are gratefully acknowledged. The usual disclaimer applies.
Submitted: Mar 26 2013. **Revised:** November 15, 2013.

Mathematical Appendix

"How to Add Apples and Pears: Non-Symmetric Nash Bargaining and the Generalized Joint Surplus"

Samuel Danthine and Noemí Navarro

Lemma 1 *Let the functions a and b be concave in $\Re^m$. Then the function f is concave in $a(\Re^m)$.*

Proof of Lemma 1. We show that for any u and $\tilde{u}$ in $a(\Re^m)$ and any t in $[0, 1]$ we have that $f(tu + (1-t)\tilde{u}) \geq tf(u) + (1-t)f(\tilde{u})$. Let us denote by $x(u)$ and $x(\tilde{u})$ the solutions of the constraint maximization problem in (2) where the level of $a(x)$ is fixed equal to u and $\tilde{u}$, respectively. Since the constraint is binding, it has to be that

$$a(x(u)) = u \text{ and } a(x(\tilde{u})) = \tilde{u}.$$

Recall that function a is concave, which means that

$$tu + (1-t)\tilde{u} = ta(x(u)) + (1-t)a(x(\tilde{u})) \leq a\left(tx(u) + (1-t)x(\tilde{u})\right).$$

This indicates that the vector $tx(u) + (1-t)x(\tilde{u})$ in $\Re^m$ belongs to the set $\{x \in \Re^m$ such that $b(x) \geq tu + (1-t)\tilde{u}\}$, or, alternatively, satisfies the constraint in the maximization problem (2) where the utility level of a is being fixed at $tu + (1-t)\tilde{u}$. Hence, $f(tu + (1-t)\tilde{u}) \geq b\left(tx(u) + (1-t)x(\tilde{u})\right)$ as it is the value function of the constraint maximization problem (2) where the utility level of a is being fixed at $tu + (1-t)\tilde{u}$. Given that b is also a concave function,

$$b\left(tx(u) + (1-t)x(\tilde{u})\right) \geq tb(x(u)) + (1-t)b(x(\tilde{u})) = tf(u) + (1-t)f(\tilde{u}),$$

as, by definition, $x(u)$ and $x(\tilde{u})$ maximize b subject to the corresponding constraints, i.e., $b(x(u)) = f(u)$ and $b(x(\tilde{u})) = f(\tilde{u})$. Hence, $f(tu + (1-t)\tilde{u}) \geq tf(u) + (1-t)f(\tilde{u})$. ■

Lemma 2 *Let $f(u)$ be, as defined before, the value function of the maximization problem in (2), with $u \in a(\Re^m)$. If f is twice differentiable with $f'(u) < 0$ and $f''(u) \leq 0$ for all u, then the generalized Nash product, $N(u) = (u - d_A)^{\alpha}\left(f(u) - d_B\right)^{1-\alpha}$, as a function of u, is strictly concave in $a(\Re^m)$.*

Proof of Lemma 2. Taking the derivative of $N(u)$ with respect to u,

$$N'(u) = N(u)\left[\frac{\alpha}{u-d_A} + \frac{(1-\alpha)\, f'(u)}{f(u)-d_B}\right],$$

and

$$N''(u) = N(u)\left\{\left[\frac{\alpha}{u-d_A} + \frac{(1-\alpha)\, f'(u)}{f(u)-d_B}\right]^2 - \frac{\alpha}{(u-d_A)^2} - \frac{(1-\alpha)\,[f'(u)]^2}{[f(u)-d_B]^2} + \frac{(1-\alpha)\, f''(u)}{f(u)-d_B}\right\}.$$

Rearranging terms,

$$N''(u) = (1-\alpha)N(u)\left\{\frac{f''(u)}{f(u)-d_B} + \frac{2\alpha f'(u)}{(u-d_A)\,(f(u)-d_B)} - \alpha\left[\frac{1}{(u-d_A)^2} + \frac{(f'(u))^2}{(f(u)-d_B)^2}\right]\right\}.$$

Since $f'(u) < 0$, $\alpha \in (0,1)$, and $f''(u) \leq 0$ we know that $N''(u) < 0$. Hence, the generalized Nash product is a strictly concave function in u. ■

Proof of Proposition 1

By Lemma 2, if $f''(u) \leq 0$ then any u^* satisfying the first-order condition of the maximization of $N(u)$ is a maximizer (and not a minimizer). Furthermore, we know by essentiality of S that the optimal solution u^* to the maximization problem satisfies that $u^* > d_A$ and $f(u^*) > d_B$, and that it belongs to the frontier F. These two aspects imply two things. First, the first order condition of the maximization problem has to be satisfied with equality. Hence, u^* is a non-symmetric Nash bargaining solution if and only if u^* solves:

$$(u^* - d_A)^{\alpha}\,(f(u^*) - d_B)^{1-\alpha}\left(\frac{\alpha}{u^*-d_A} + (1-\alpha)\frac{f'(u^*)}{f(u^*)-d_B}\right) = 0. \tag{A1}$$

Second, since the non-symmetric Nash bargaining solution belongs to the frontier F we know there is an $x^* \in \Re^m$ such that $u^* = a(x^*)$ and $f(u^*) = b(x^*)$. Recall then that, for any issue i, $f'(u^*) = \frac{b_i(x^*)}{a_i(x^*)}$, where $a(x^*) = u^* > d_A$, given that x^* solves the maximization problem in (2), and by the envelope theorem, the derivative of the value function with respect to u, whenever it exists, is equal to the derivative of the Lagrangian function associated to the problem in (2) with respect to u. With all this we can rewrite the first order condition in (A1) as:

$$(a(x^*) - d_A)^\alpha (b(x^*) - d_B)^{1-\alpha} \left(\alpha \frac{a_i(x^*)}{a(x^*) - d_A} + (1-\alpha) \frac{b_i(x^*)}{b(x^*) - d_B}\right) = 0, \qquad \text{(A2)}$$

for any issue i. By essentiality of our bargaining problem there is at least one $x \in \Re^m$ such that $a(x) > d_A$ and $b(x) > d_B$. This implies that $u^* > d_A$ and $f(u^*) > d_B$, otherwise u^* cannot maximize the generalized Nash product. All this indicates that equation (A2) is true if and only if:

$$\alpha a_i(x^*) (b(x^*) - d_B) + (1-\alpha) b_i(x^*) (a(x^*) - d_A) = 0,$$

for any issue i. Rearranging terms, we obtain the formula in Proposition 1. ■