Economics Bulletin

Appendix and Supplemental material not intended for publication-Round 1

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Appendices

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Optimal storage under uncertainty: investigating the implications of frugality and prudence
Appendices

A- Proof of Theorem 1

To prove Theorem 1, I first solve the problem recursively.

$$\max_{\{q_1,y_1\}} U(q_1) - C(y_1)$$

subject to Eqs. (1a), $q_1 \geq 0$ and $y_1 \geq 0$. The first-order necessary condition for a maximum yields:[1]

$$U'(y_1 + z_1 + s) - C'(y_1) \leq 0,$$

with equality for $y_1 > 0$. Let $y_1 \equiv y_1(z_1 + s)$. Given y_1, the maximum value function for period 1 is

$$W_1(z_1, s) = U(y_1 + z_1 + s) - C(y_1). \tag{1}$$

The problem in period 0 is

$$\max_{\{q_0,y_0,s\}} U(q_0) - C(y_0) + \mathbb{E}\left[W_1(\tilde{z}_1, s)\right]$$

subject to (1a), and $q_0 \geq 0$, $y_0 \geq 0$, $\bar{s} \geq s$, and $s \geq 0$. The first-order necessary condition for dispatchable generation at a maximum is[2]

$$U'(y_0 + z_0 - \alpha s) - C'(y_0) \leq 0,$$

with equality for $y_0 > 0$. Let $y_0 \equiv y_0(z_0 - \alpha s)$.

Using the maximum value function in Eq. (1) and the Envelope Theorem, the first-order condition with respect to s is

$$\begin{array}{ll} \mathbb{E}\left[U'(\tilde{y}_1 + \tilde{z}_1)\right] - \alpha U'(y_0 + z_0) \leq 0 & \text{if } s = 0, \\ \mathbb{E}\left[U'(\tilde{y}_1 + \tilde{z}_1 + s)\right] - \alpha U'(y_0 + z_0 - \alpha s) = 0 & \text{if } s \in (0, \min(\bar{s}, k)), \\ \mathbb{E}\left[U'(\tilde{y}_1 + \tilde{z}_1 + \min(\bar{s}, k))\right] - \alpha U'(y_0 + z_0 - \alpha \min(\bar{s}, k)) \geq 0 & \text{otherwise, (i.e., } s = \min(\bar{s}, k)\text{),} \end{array}$$

[1] The second-order condition for a maximum is satisfied by $U''(q_1) - C''(y_1) < 0$.

[2] Similar to the problem in the final period, the second-order condition for a maximum is satisfied: $U''(q_0) - C''(y_0) < 0$.

where $k \equiv (y_0 + z_0)/\alpha$.[3] Since $U'(0) > C'(0)$, $y_0 > 0$ if $z_0 = s = 0$. Thus, $q_0 = y_0 > 0$. $s \in (0, \min(\bar{s}, k)]$ ensures that q_0 is non-negative.

Let

$$s^+ = \arg\max_s \; U(y_0(z_0 - \alpha s) + z_0 - \alpha s) - C(y_0(z_0 - \alpha s)) + W_1(\mu, s)$$

and

$$s = \arg\max_s \; U(y_0(z_0 - \alpha s) + z_0 - \alpha s) - C(y_0(z_0 - \alpha s)) + \mathbb{E}[W_1(\tilde{z}_1, s)]$$

Proof. Assume s and s^+ are interior. s^+ satisfies

$$-U'(y_0(z_0 - \alpha s^+) + z_0 - \alpha s^+) + \frac{1}{\alpha}[U'(y_1(\mu + s^+) + \mu + s^+)] = 0,$$

while s satisfies

$$-U'(y_0(z_0 - \alpha s) + z_0 - \alpha s) + \frac{1}{\alpha}\mathbb{E}[U'(y_1(\tilde{z}_1 + s) + \tilde{z}_1 + s)] = 0,$$

where $\tilde{z}_1 = \mu + \tilde{\varepsilon}$ with $\mathbb{E}[\tilde{\varepsilon}] = 0$.

Given the concavity of the problem (i.e., U is a concave function and C is a convex function), $s \geq s^+$ if and only if

$$-U'(y_0(z_0 - \alpha s^+) + z_0 - \alpha s^+) + \frac{1}{\alpha}\mathbb{E}[U'(y_1(\mu + \tilde{\varepsilon} + s^+) + \mu + \tilde{\varepsilon} + s^+)] \geq 0.$$

This condition comes down to

$$\mathbb{E}[U'(y_1(\mu + \tilde{\varepsilon} + s^+) + \mu + \tilde{\varepsilon} + s^+)] \geq U'(y_1(\mu + s^+) + \mu + s^+). \tag{2}$$

From Jensen's inequality, Eq. (2) holds if and only if

$$z \mapsto U'(y_1(z + s) + z + s)$$

is a convex function:

$$\frac{\partial^2 U'(q_1)}{\partial z^2} = U'''(q_1)\Big(\frac{\partial y_1}{\partial z} + 1\Big)^2 + U''(q_1)\frac{\partial^2 y_1}{\partial z^2} \geq 0. \tag{3}$$

Observe that

$$\frac{\partial y_1}{\partial z} = \frac{U''(q_1)}{C''(y_1) - U''(q_1)} < 0, \tag{4}$$

and

$$\frac{\partial^2 y_1}{\partial z^2} = \frac{\big(C''(y_1)\big)^2 U'''(q_1) - \big(U''(q_1)\big)^2 C'''(y_1)}{\big(C''(y_1) - U''(q_1)\big)^3}, \tag{5}$$

[3] The second-order condition for a maximum gives $\alpha^2 U''(q_0) + \mathbb{E}[U''(q_1)] < 0$.

Substituting (4) and (5) in (3) gives

(6)
$$\frac{(C''(y_1))^3}{\big(C''(y_1)-U''(q_1)\big)^3}U'''(q_1)+\frac{(-U''(q_1))^3}{\big(C''(y_1)-U''(q_1)\big)^3}C'''(y_1)\geq 0.$$

Consider the corner solutions. Suppose $s^+ = \min(\bar{s}, k)$ ($k \equiv (y_0 + z_0)/\alpha$). Following the same steps as before, it can be shown that

$$\mathbb{E}[U'(y_1(\mu+\tilde{\varepsilon}+\min(\bar{s},k))+\mu+\tilde{\varepsilon}+\min(\bar{s},k))]\geq U'(y_1(\mu+\min(\bar{s},k))+\mu+\min(\bar{s},k))$$

if and only if (6) holds. But $s^+ = \min(\bar{s}, k)$. Therefore, $s^* = s^+ = \min(\bar{s}, k)$.

Suppose now $s^+ = 0$. Given $s^+ = 0$,

$$\mathbb{E}[U'(y_1(\mu+\tilde{\varepsilon})+\mu+\tilde{\varepsilon})]\geq U'(y_1(\mu)+\mu)$$

if and only if (6) holds. Therefore, $s^* \geq s^+ = 0$.

This completes the proof of the Theorem 1. □

B- Comparative statics

Theorem 1 indicates that $s \geq s^+$ if and only if (6) is satisfied. Given that

$$\frac{\partial y_0}{\partial s}=\frac{-\alpha U''(q_0)}{C''(y_0)-U''(q_0)}>0$$

and $q_0 = y_0 + z_0 - \alpha s$, I can calculate

$$\frac{\partial q_0}{\partial s}=\frac{-\alpha C''(y_0)}{C''(y_0)-U''(q_0)}<0.$$

Furthermore, since

$$\frac{\partial y_1}{\partial s}=\frac{U''(q_1)}{C''(y_1)-U''(q_1)}<0,$$

and $q_1 = y_1 + z_1 + s$, I get

$$\frac{\partial q_1}{\partial s}=\frac{C''(y_1)}{C''(y_1)-U''(q_1)}>0.$$