Economics Bulletin

Appendix and Supplemental material not intended for publication-Round 2

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Appendices: Comparison of the GBM and VG process densities, and ADF test and autocorrelation function.

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A The geometric Brownian motion process

In this section, we sketch the main characteristics of the GBM process. Let $\{P_t\}_{t\geq 0}$ be an asset price at time t. The GBM is defined as:

$$dP_t = \mu P_t dt + \sigma P_t W_t, \tag{5}$$

where $P_0 > 0$. In equation (5), $\mu \in \mathbb{R}$ is the drift parameter, and $\sigma > 0$ measures volatility. Additionally, W_t is the standard increment of a Wiener process.

Now, define $y_t = \ln(P_t)$. Applying Ito's lemma and using equation (5), we get:

$$dy_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma W_t. \tag{6}$$

Equation (6) implies that y_t follows an arithmetic Brownian process with drift equal to $\left(\mu - \frac{1}{2}\sigma^2\right)$ and volatility σ. By choosing a discrete time interval $\Delta t = t - q$ with $q < t$, and letting $X_t = y_t - y_q$ be the log price increments (continuously compounded returns) over a time period of Δt, we can derive a discrete-time version of equation (6). This is:

$$X_t = \left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + (W_t - W_q)\sigma. \tag{7}$$

The properties of the standard Brownian motion let us rewrite equation (7) as[7]

$$X_t = \left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + \sigma\varepsilon\sqrt{\Delta t}, \quad \varepsilon \sim \mathcal{N}(0,1).$$

Then, it is easy to conclude that

$$X_t \sim \mathcal{N}\left(\left[\mu - \frac{1}{2}\sigma^2\right] \Delta t, \Delta t \sigma\right), \tag{8}$$

which implies that both the mean and volatility of X_t increase proportionally to the length of time over which the asset is held.

The parameters of a GBM process can easily be found by maximum likelihood estimation. By fixing $\Delta t = 1$, and taking into account that the log returns follow a normal distribution, we can compute the sample mean and variance as

$$\overline{x} = \frac{1}{n}\sum_{t=1}^{T} x_i, \quad s^2 = \frac{1}{n-1}\sum_{t=1}^{T}(x_i - \overline{x})^2.$$

Hence, from equation (8),

$$\widehat{\sigma}^2 = s^2, \quad \widehat{\mu} = \overline{x} + \frac{s^2}{2}. \tag{9}$$

[7] See Venegas-Martínez (2006), for instance.

B Comparison of the GBM and VG process densities

We show in Figure 3 how GBM and VG process densities compare to empirical density function. As we can see, none of the distributions adjust well the peak of the empirical distribution, but the variance-gamma density function is closer to the empirical mean value. It is also true that the variance-gamma distribution adjusts better in both tales compared to normal distribution. This is also evident from the log-density function in Figure 3.

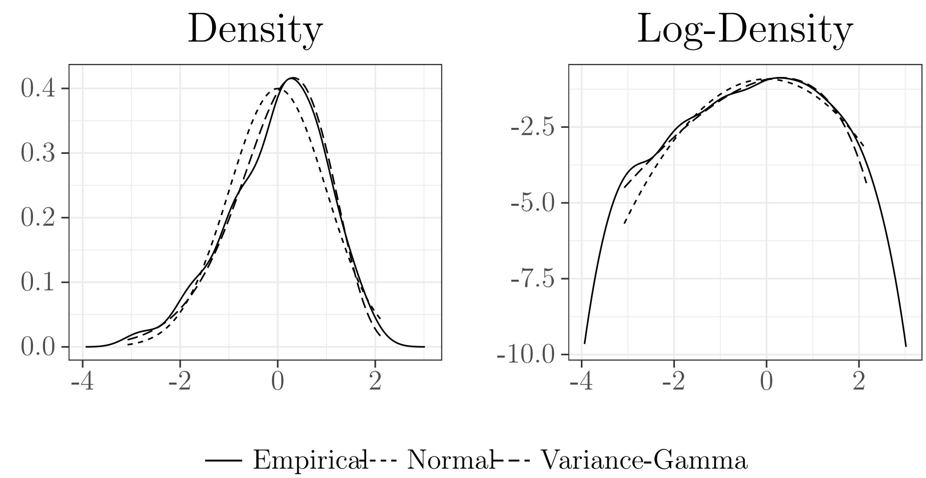

Figure 3: Empirical density function of fuel energy index log returns compared to estimated densities.

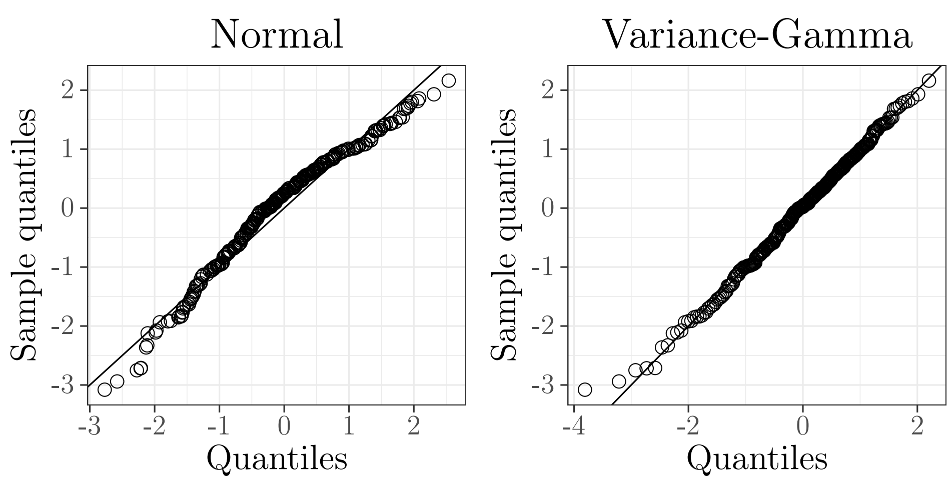

Figure 4: QQ-Plots, normal and variance-gamma distributions.

We can also see the performance of both processes by looking at the qq-plots in Figure 4. Again, it is clear that the variance-gamma distribution shows a better fit to data. This is particularly true for higher quantiles.

C Augmented Dickey-Fuller Test and Autocorrelation Functions

We show in Table IV that the IMF's fuel energy index is stationary as the hyphotesis of unit root is always rejected.

Table IV: p-values for augmented Dickey-Fuller test for several lags.

	p-values	
Lag	*Actual Returns*	*Filtered Returns*
1	<0.01	<0.01
2	<0.01	<0.01
3	<0.01	<0.01
4	<0.01	<0.01
5	<0.01	<0.01
6	<0.01	<0.01

In Figure 5 we show the autocorrelation function for both X_t^2 and $\widehat{Y}_t^2$. We can see that the former is autocorrelated and the later is not.

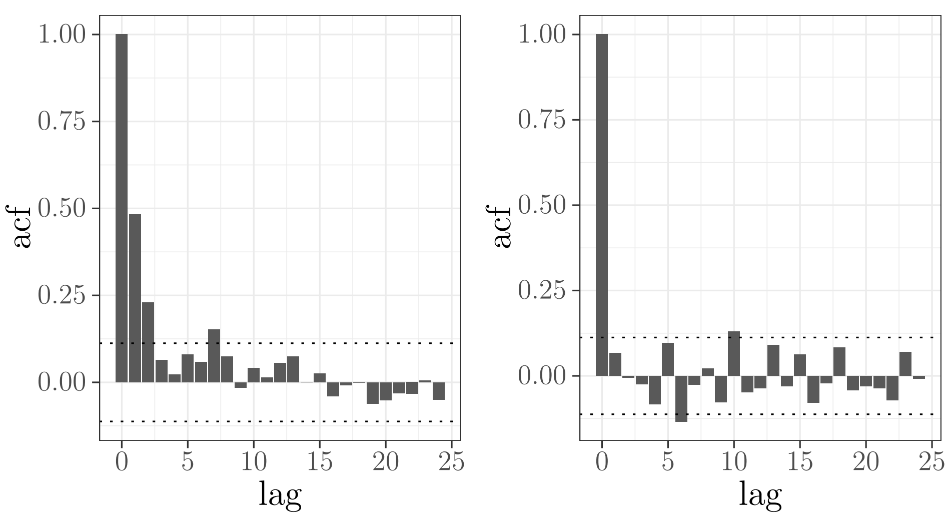

Figure 5: Autocorrelation function for the squared log-returns of actual data (left) and the squared returns of filtered data (right).