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Appendix

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Appendix

Proof of Proposition 1. Let $\Gamma < (1+r)W_1$. Then there are two cases that can occur
(P1): $W_{2b} \geq \Gamma,\ W_{2g} \geq \Gamma$ for $0 \leq \alpha \leq \frac{(1+r)W_1-\Gamma}{(r-x_b)W_1}$
(P2): $W_{2b} < \Gamma,\ W_{2g} \geq \Gamma$ for $\frac{(1+r)W_1-\Gamma}{(r-x_b)W_1} < \alpha \leq \alpha_U$
The corresponding problems are

$$\left.\begin{aligned}\text{Max}_\alpha: \quad & \mathbb{E}(U_{LA}(W_2-\Gamma) = pU_G(W_{2g}-\Gamma) + (1-p)U_G(W_{2b}-\Gamma) \\ & = \tfrac{1}{1-\gamma}\{p[(x_g-r)W_1\alpha + (1+r)W_1 - \Gamma]^{1-\gamma} \\ & +(1-p)[-(r-x_b)W_1\alpha + (1+r)W_1 - \Gamma]^{1-\gamma}\} \\ \text{such that}: \quad & 0 \leq \alpha \leq \tfrac{(1+r)W_1-\Gamma}{(r-x_b)W_1}\end{aligned}\right\} \qquad \text{(P1)}$$

$$\left.\begin{aligned}\text{Max}_\alpha: \quad & \mathbb{E}(U_{LA}(W_2-\Gamma) = pU_G(W_{2g}-\Gamma) + (1-p)\lambda U_L(W_{2b}-\Gamma) \\ & = \tfrac{1}{1-\gamma}\{p[(x_g-r)W_1\alpha + (1+r)W_1 - \Gamma]^{1-\gamma} \\ & -\lambda(1-p)[(r-x_b)W_1\alpha - (1+r)W_1 + \Gamma]^{1-\gamma}\} \\ \text{such that}: \quad & \tfrac{(1+r)W_1-\Gamma}{(r-x_b)W_1} < \alpha \leq \alpha_U\end{aligned}\right\} \qquad \text{(P2)}$$

The idea of the proof is to show that (P1) is a concave programming problem with the global maximum being α^* as defined by (4) which is such that $0 \leq \alpha^* \leq \frac{(1+r)W_1-\Gamma}{(r-x_b)W_1}$ and the utility of (P2) is decreasing for $\lambda > 1/K_\gamma$ that is the assumption of Proposition 1. The first order conditions (FOC) for (P1) are

$$p[(x_g-r)W_1\alpha+(1+r)W_1-\Gamma]^{-\gamma}(x_g-r)-(1-p)[-(r-x_b)W_1\alpha+(1+r)W_1-\Gamma]^{-\gamma}(r-x_b)=0$$

or alternatively

$$\frac{p}{1-p}\left[\frac{-(r-x_b)W_1\alpha + (1+r)W_1 - \Gamma}{(x_g-r)W_1\alpha + (1+r)W_1 - \Gamma}\right]^\gamma = \frac{r-x_b}{x_g-r}$$

where the left hand side, as shown below, is the MRS and the right hand side is the market trade-off as given by its definition. It can be shown that the second order conditions are satisfied which implies that (P1) is a concave problem. Finally, it can be seen that

$$\alpha^* = \frac{\left(1-K_0^{1/\gamma}\right)}{r-x_b+K_0^{1/\gamma}(x_g-r)}\frac{(1+r)W_1-\Gamma}{W_1}$$

satisfies the FOC and that $0 \leq \alpha^* \leq \frac{(1+r)W_1-\Gamma}{(r-x_b)W_1}$. Note that $\alpha^* > 0$ follows from assumption $\mathbb{E}(x) > r$ which implies that $K_0 < 1$.

Regarding problem (P2), note that

$$\begin{aligned}1/K_\gamma &= \frac{p(x_g-r)^{1-\gamma}}{(1-p)(r-x_b)^{1-\gamma}} = \frac{p[(r-x_b)W_1\alpha]^\gamma(x_g-r)}{(1-p)[(x_g-r)W_1\alpha]^\gamma(r-x_b)} \\ &> \frac{p[(r-x_b)W_1\alpha-(1+r)W_1+\Gamma]^\gamma(x_g-r)}{(1-p)[(x_g-r)W_1\alpha+(1+r)W_1-\Gamma]^\gamma(r-x_b)}\end{aligned}$$

Thus, based on this and $\lambda > 1/K_\gamma$ it follows that

$$\lambda > \frac{p[(r-x_b)W_1\alpha-(1+r)W_1+\Gamma]^\gamma(x_g-r)}{(1-p)[(x_g-r)W_1\alpha+(1+r)W_1-\Gamma]^\gamma(r-x_b)}$$

and consequently

$$\frac{d\mathbb{E}(U_{LA}(W_2-\Gamma_2))}{d\alpha} < 0$$

implying that the utility of (P2) is a decreasing function in α.

The MRS for (P1) can be obtained as follows: We derive $d\mathbb{E}(U_{LA}(W_2-\Gamma)$ and from equation $d\mathbb{E}(U_{LA}(W_2-\Gamma)=0$ express the MRS. Thus,

$$\begin{aligned}d\mathbb{E}(U_{LA}(W_2-\Gamma) &= p\left[(x_g-r)W_1\alpha+(1+r)W_1-\Gamma\right]^{-\gamma}d(W_{2g}-\Gamma) \\ &\quad +(1-p)\left[-(r-x_b)W_1\alpha+(1+r)W_1-\Gamma\right]^{-\gamma}d(W_{2b}-\Gamma) \\ &= 0\end{aligned}$$

This implies that for (P1)

$$\left.\frac{d(W_{2b}-\Gamma)}{d(W_{2g}-\Gamma)}\right|_{\alpha\in\left[0,\frac{(1+r)W_1-\Gamma}{(r-x_b)W_1}\right]} = -\frac{p}{1-p}\left[\frac{-(r-x_b)W_1\alpha+(1+r)W_1-\Gamma}{(x_g-r)W_1\alpha+(1+r)W_1-\Gamma}\right]^\gamma$$

■

Proof of Proposition 2. At first we re-formulate the statement of Proposition 2 in more detail where the MRS is considered for $\lambda \in (0, \alpha_U]$.

Let $\Gamma = (1+r)W_1$. Then problem (2) obtains its maximum (maxima) α^* as follows

(i) $\alpha^* = \alpha_U$ *for* $MRS > \frac{r-x_b}{x_g-r}$ *and* $p > p_\gamma$

(ii) $\alpha^* \in [0, \alpha_U]$ *for* $MRS = \frac{r-x_b}{x_g-r}$ *and* $p > p_\gamma$

(iii) $\alpha^* = 0$ *for* $MRS < \frac{r-x_b}{x_g-r}$

Note that $\alpha^* > 0$ for $MRS > \frac{r-x_b}{x_g-r}$ and $p > p_\gamma$.

Based on the domain of α, there is only one case that can occur: $0 \le W_{2b} \le \Gamma$ and $W_{2g} \ge \Gamma$. Thus, the corresponding problem we would like to solve is

$$\left.\begin{aligned}\text{Max}_\alpha:\ & pU_G(W_{2g}-\Gamma)+(1-p)\lambda U_L(W_{2b}-\Gamma) = \\ & \tfrac{(W_1\alpha)^{1-\gamma}}{1-\gamma}\left[p(x_g-r)^{1-\gamma}-\lambda(1-p)(r-x_b)^{1-\gamma}\right] \\ \text{such that}:\ & 0 \le \alpha \le \alpha_U\end{aligned}\right\} \qquad \text{(P2)}$$

The MRS for (P2) can be obtained as follows: We derive $d\mathbb{E}(U_{LA}(W_2-\Gamma)$ and from equation $d\mathbb{E}(U_{LA}(W_2-\Gamma)=0$ express the MRS. Thus,

$$d\mathbb{E}(U_{LA}(W_2-\Gamma)=p\left[(x_g-r)W_1\alpha\right]^{-\gamma}d(W_{2g}-\Gamma)+\lambda(1-p)\left[-(r-x_b)W_1\alpha\right]^{-\gamma}d(W_{2b}-\Gamma)=0$$

This implies that for (P2)

$$\left.\frac{d(W_{2b}-\Gamma)}{d(W_{2g}-\Gamma)}\right|_{\alpha\in[0,\alpha_U]}=-\frac{p}{1-p}\left[\frac{r-x_b}{x_g-r}\right]^{\gamma}\frac{1}{\lambda}$$

and thus the $MRS=\frac{p}{1-p}\left[\frac{r-x_b}{x_g-r}\right]^{\gamma}\frac{1}{\lambda}$.

Case (i): Note that $MRS>\frac{r-x_b}{x_g-r}$ implies that

$$p(x_g-r)^{1-\gamma}-\lambda(1-p)(r-x_b)^{1-\gamma}>0 \tag{A1}$$

and thus the objective function of problem $(P2)$ is increasing at its domain. Based on this and the fact that objective function of $(P2)$ is zero for $\alpha=0$ it follows that in case (i) the objective function $\mathbb{E}(U_{LA}(W_2-\Gamma))$ is increasing in its domain and thus $\alpha^*=\alpha_U$. Note that $MRS>\frac{r-x_b}{x_g-r}$ can hold only when $1<\lambda<1/K_\gamma$ and $p>p_\gamma$.

Case (ii): Note that $MRS=\frac{r-x_b}{x_g-r}$ implies that

$$p(x_g-r)^{1-\gamma}-\lambda(1-p)(r-x_b)^{1-\gamma}=0$$

and thus the objective function of $(P2)$ is constant (namely zero) in its domain. Thus, in case (ii) the objective function $\mathbb{E}(U_{LA}(W_2-\Gamma))$ has its maxima in $[0,\alpha_U]$.

Case (iii): Note that $MRS<\frac{r-x_b}{x_g-r}$ implies that

$$p(x_g-r)^{1-\gamma}-\lambda(1-p)(r-x_b)^{1-\gamma}<0$$

and thus the objective function of $(P2)$ is decreasing at its domain. In summary, the objective function $\mathbb{E}(U_{LA}(W_2-\Gamma))$ has its maxima at zero, i.e., $\alpha^*=0$.

Note finally that cases (A1) and (A2) can happen for $\lambda>1$ only when $K_\gamma<1$ and thus when $p>p_\gamma$. ■

Proof of Proposition 3. Let $\Gamma>(1+r)W_1$. Then there are again two cases that can occur

(P3) $W_{2b}<\Gamma,\ W_{2g}<\Gamma$ for $0\le\alpha\le\min\left\{\frac{\Gamma-(1+r)W_1}{(x_g-r)W_1},\alpha_U\right\}$

(P4) $W_{2b}<\Gamma,\ W_{2g}\ge\Gamma$ for $\frac{\Gamma-(1+r)W_1}{(x_g-r)W_1}<\alpha\le\alpha_U$ if $\Gamma\le(1+r)W_1\frac{x_g-x_b}{r-x_b}$

and thus the corresponding problems are

$$\left.\begin{aligned}\text{Max}_\alpha: \quad & p\lambda U_L(W_{2g}-\Gamma)+(1-p)\lambda U_L(W_{2b}-\Gamma)\\ = \quad & -\tfrac{1}{1-\gamma}\lambda p[-(x_g-r)W_1\alpha+\Gamma-(1+r)W_1]^{1-\gamma}+\\ & -\tfrac{1}{1-\gamma}\lambda(1-p)[(r-x_b)W_1\alpha+\Gamma-(1+r)W_1]^{1-\gamma}\\ \text{such that}: \quad & 0\le\alpha\le\min\left\{\tfrac{\Gamma-(1+r)W_1}{(x_g-r)W_1},\alpha_U\right\}\end{aligned}\right\} \tag{P3}$$

$$\left.\begin{aligned}\text{Max}_\alpha: \quad & pU_G(W_{2g}-\Gamma)+(1-p)\lambda U_L(W_{2b}-\Gamma)\\ = \quad & \tfrac{1}{1-\gamma}p[(x_g-r)W_1\alpha+(1+r)W_1-\Gamma]^{1-\gamma}\\ & -\tfrac{1}{1-\gamma}\lambda(1-p)[(r-x_b)W_1\alpha+\Gamma-(1+r)W_1]^{1-\gamma}\\ \text{such that}: \quad & \tfrac{\Gamma-(1+r)W_1}{(x_g-r)W_1}<\alpha\le\alpha_U \quad \text{and if} \quad \Gamma\le(1+r)W_1\tfrac{x_g-x_b}{r-x_b}\end{aligned}\right\} \tag{P4}$$

Note that (P4) is the same as (P2) except for a different set of feasible solutions. (P3) is a convex programming problem (in α) and thus its maximum will be reached at one of the end points. The first order conditions for (P4) are

$$p[(x_g-r)W_1\alpha+(1+r)W_1-\Gamma]^{-\gamma}(x_g-r)-\lambda(1-p)[(r-x_b)W_1\alpha-(1+r)W_1+\Gamma]^{-\gamma}(r-x_b)=0$$

or alternatively

$$\frac{p}{\lambda(1-p)}\left[\frac{(r-x_b)W_1\alpha-(1+r)W_1+\Gamma}{(x_g-r)W_1\alpha+(1+r)W_1-\Gamma}\right]^\gamma=\frac{r-x_b}{x_g-r}$$

where the left hand side, as shown below, is the MRS and the right hand side is the market trade-off. It can be easily verified that α^* given by (8) satisfies the FOC. In addition

$$\frac{d\mathbb{E}(U_{LA}(W_2-\Gamma))}{d\alpha}>0 \quad \text{for} \quad \alpha<\alpha^* \tag{A2}$$

$$\frac{d\mathbb{E}(U_{LA}(W_2-\Gamma))}{d\alpha}<0 \quad \text{for} \quad \alpha>\alpha^* \tag{A3}$$

which implies that $\mathbb{E}(U_{LA}(W_2-\Gamma))$ reaches its maximum at α^*. In more detail, if $\frac{d\mathbb{E}(U_{LA}(W_2-\Gamma))}{d\alpha}>0$ then

$$(p(x_g-r))^{\frac{1}{\gamma}}[(r-x_b)W_1\alpha-(1+r)W_1+\Gamma]>(\lambda(1-p)(r-x_b))^{\frac{1}{\gamma}}[(x_g-r)W_1\alpha+(1+r)W_1-\Gamma]$$

and simple derivations show that the last inequality follows from $\alpha<\alpha^*$ and $\lambda>\frac{1}{K_\gamma}$ which thus proves (A2). Inequality (A3) can be shown in a similar matter. It can be easily checked that $\alpha^*<\alpha_U$ for $\Gamma<\Gamma_U$ and thus $\alpha^*=\alpha_U$ for $\Gamma\ge\Gamma_U$.

To calculate the MRS for (P4) we again derive $d\mathbb{E}(U_{LA}(W_2 - \Gamma)$ and from equation $d\mathbb{E}(U_{LA}(W_2 - \Gamma) = 0$ express the MRS, i.e.,

$$\begin{aligned} d\mathbb{E}(U_{LA}(W_2 - \Gamma) &= p\left[(x_g - r)W_1\alpha + (1+r)W_1 - \Gamma\right]^{-\gamma} d(W_{2g} - \Gamma) \\ &\quad -\lambda(1-p)\left[(r - x_b)W_1\alpha - (1+r)W_1 + \Gamma\right]^{-\gamma} d(\Gamma - W_{2b}) \\ &= 0 \end{aligned}$$

which implies

$$\left.\frac{d(\Gamma - W_{2b})}{d(W_{2g} - \Gamma)}\right|_{\frac{\Gamma-(1+r)W_1}{(x_g - r)W_1} < \alpha \leq \alpha_U} = \frac{p}{1-p}\left[\frac{(r - x_b)W_1\alpha - (1+r)W_1 + \Gamma}{(x_g - r)W_1\alpha + (1+r)W_1 - \Gamma}\right]^{\gamma}\frac{1}{\lambda}$$

■