Appendix A: Proofs

Because all proofs are based on the same underlying idea, we start by stating some general remarks that will be useful in all of the proofs.

Note first that \( t_r > b > t_l > b \) for all \( b \in [b, b^*] \) since capacity \( c \) is exceeded for all \( b \in [b, b^*] \), and \( t_l = t_r = b > b \) for \( b = b^* \) since the “height” of the peak equals \( c \) for \( b = b^* \). Hence, the welfare function is for any \( b \in [b, b^*] \) given by:

\[
W(b) = \lambda \int_0^T y(t \mid b)dt + (1 - \lambda) \int_0^T h(t \mid b)dt,
\]

\[
= \lambda \int_0^T (1 - g(b)\dot{x}(t))dt + (1 - \lambda) \left( \int_{t_l}^{t_l} 1dt + \int_{t_l}^{t_r} (1 - (\dot{x}(t) - c))dt + \int_{t_r}^T 1dt \right),
\]

\[
= \lambda \left[ t - g(b)x(t) \right]_0^T + (1 - \lambda) \left( [t]_{t_l}^{t_l} + [t + ct - x(t)]_{t_l}^{t_r} + [t]_{t_r}^T \right),
\]

\[
= T - \lambda g(b)(x(T) - x(0)) + (1 - \lambda)(c(t_r - t_l) - (x(t_r) - x(t_l))). \tag{10}
\]

Note also that if \( b \in [b, b^*] \), it follows that:

\[
c(t_r - t_l) \leq x(t_r) - x(t_l), \tag{11}
\]

with strict inequality for \( b \in [b, b^*] \). Finally, if \( b > b^* \) the capacity \( c \) is never exceeded so the welfare function (10) can be simplified to:

\[
W(b) = T - \lambda g(b)(x(T) - x(0)). \tag{12}
\]

Proof of Proposition 1. Consider first part (i), and suppose that \( b \in [b, b^*] \). By the above conclusions, it follows that \( t_r > b > t_l > b \) for all \( b \in [b, b^*] \), and \( t_l = t_r = b > b \) for \( b = b^* \). Furthermore, since \( \lambda = 0 \) the welfare function (10) can be written as:

\[
W(b) = T - (c(t_r - t_l) - (x(t_r) - x(t_l))).
\]

From equation (11), it follows that \( W(b) < T \) for any \( b \in [b, b^*] \) and \( W(b) = T \) for \( b = b^* \). If, on the other hand, \( b > b^* \) and \( \lambda = 0 \), the capacity is never exceeded so \( W(b) = T \) by equation (12). This proves part (i) of the proposition.

To prove part (ii), note that since \( \lambda = 1 \), the welfare function (10) reduces to:

\[
W(b) = T - g(b)(x(T) - x(0)). \tag{13}
\]
Since \( g(b) \geq 1 \) and \( g'(b) > 0 \) for all \( b \geq b_* \), and \( x(T) - x(0) > 0 \), it follows that equation (13) is maximized when \( b \) is minimized. Hence, \( b = b_* \) maximizes equation (13). \( \square \)

**Proof of Proposition 2.** Consider first the case when \( b \in [b_*, b^*] \). Because \( g(b) = 1 \) and \( g'(b) = 0 \) for all \( b \geq b_* \) by assumption, equation (10) reduces to:

\[
W(b) = T - \lambda g(b)(x(T) - x(0)) + (1 - \lambda)(c(t_r - t_t) - (x(t_r) - x(t_t))).
\]

Because \( T - \lambda(x(T) - x(0)) \) is a constant, equation (14) is, by condition (11), maximized when \( b = b^* \), i.e., when \( W(b^*) = T - \lambda(x(T) - x(0)) \). To complete the proof, we need only to demonstrate that the welfare equals \( T - \lambda(x(T) - x(0)) \) for any \( b > b^* \). But if \( b > b^* \), the capacity is never exceeded so the welfare function is given by equation (12) for \( g(b) = 1 \) and \( g'(b) = 0 \), i.e., \( W(b) = T - \lambda(x(T) - x(0)) \) for all \( b > b^* \), which concludes the proof. \( \square \)

**Proof of Proposition 3.** Consider first the case when \( b \in [b_*, b^*] \). Then the welfare function is given by equation (10), i.e.:

\[
W(b) = T - \lambda g(b)(x(T) - x(0)) + (1 - \lambda)(c(t_r - t_t) - (x(t_r) - x(t_t))).
\]

For \( b \geq b^* \), this equation reduces to \( W(b) = T - \lambda g(b)(x(T) - x(0)) \). Because \( x(T) - x(0) \) is a constant, \( g(b) \geq 1 \) and \( g'(b) > 0 \) for all \( b \geq b_* \), it then follows that \( W(b^*) > W(b) \) for any \( b > b^* \). Hence, the welfare cannot be maximized for any \( b > b^* \). \( \square \)

### Appendix B: Comparative Statics

In this appendix, it is first shown that the optimal peak of the pandemic \( b \) is weakly decreasing in the welfare weight attached to production \( \lambda \). It is then demonstrated that welfare \( W \) is strictly increasing in the capacity constraint \( c \).

#### Changing the Welfare Weight \( \lambda \)

The result will only be proved for the case when the health-care capacity is exceeded, noting that the corresponding result also holds when this is not the case (this can be proved by using similar arguments as in the below proof that welfare is strictly increasing in the capacity constraint).

We start by making three general observations that all relate to the derivative of the welfare function \( W \), as defined in equation (10), with respect to \( b \), i.e.:

\[
\frac{dW}{db} = -\lambda g'(b)(x(T) - x(0)) - \lambda g(b)\frac{dx(T)}{db} + (1 - \lambda)c\frac{d}{db}(t_r - t_t) - (1 - \lambda)\frac{d}{db}(x(t_r) - x(t_t)).
\]

First, note that:

\[
\frac{dt_l}{db} = \frac{dt_r}{db} = 1 \quad \text{and} \quad \frac{d}{db}(t_r - t_t) = 0.
\]
Second, by letting \( t_j = t_i \) or \( t_j = t_r \) and by using conditions (5)–(6), it follows that:
\[
\frac{dx(t_j)}{db} = -\frac{1 * a * e^{a t_j} (e^{ab} + e^{a t_j}) - e^{a t_j} (a * e^{ab} + 1 * a * e^{a t_j})}{(e^{ab} + e^{a t_j})^2} = 0.
\]

Third, note that \( x(0) \) is a constant and, consequently, that:
\[
\frac{d(x(T) - x(0))}{db} = \frac{dx(T)}{db} = -\frac{ae^{a T} e^{ab}}{(e^{ab} + e^{a T})^2} < 0.
\]

The above three observations can be used to rewrite condition (15) as:
\[
\frac{dW}{db} = -\lambda g'(b)(x(T) - x(0)) - \lambda g(b) \frac{dx(T)}{db}.
\] (16)

The necessary first-order condition for \( b \) to be a welfare maximizer can thus be written as:
\[
\frac{dW}{db} = 0 \iff -\lambda g'(b)(x(T) - x(0)) - \lambda g(b) \frac{dx(T)}{db} = 0.
\] (17)

Define now the function \( f(b(\lambda), \lambda) \) by using the first-order condition in equation (17), i.e.:
\[
f(b(\lambda), \lambda) = -\lambda g'(b(\lambda))(x(T, b(\lambda), \lambda) - x(0)) - \lambda g(b) \frac{dx(T, b(\lambda), \lambda)}{db}.
\] (19)

We will use the implicit function theorem on \( f(b(\lambda), \lambda) = 0 \) to show that \( b'(\lambda) \leq 0 \). To do this, we first use the insights from equation (16) to calculate the following derivative:
\[
\frac{d^2 x(T, b(\lambda), \lambda)}{db d\lambda} = \frac{d}{d\lambda} \left(-\frac{ae^{a T} e^{ab(\lambda)}}{(e^{ab(\lambda)} + e^{a T})^2}\right) = -\frac{ae^{a T} e^{ab(\lambda)} (e^{a T} - e^{ab(\lambda)})}{(e^{ab(\lambda)} + e^{a T})^3} b'(\lambda) = -Q(b(\lambda), \lambda)b'(\lambda).
\]

The function \( Q(\cdot) \) is positive since \( T > b \). Thus, the derivative \( d^2 x(\cdot)/db d\lambda \) is negative if \( b'(\lambda) > 0 \) and positive if \( b'(\lambda) < 0 \). Next, differentiate \( f(b(\lambda), \lambda) \) with respect to \( \lambda \):
\[
\frac{df(b(\lambda), \lambda)}{d\lambda} = -g'(b(\lambda)) (x(T, b(\lambda), \lambda) - x(0)) - \lambda g''(b(\lambda)) (x(T, b(\lambda), \lambda) - x(0)) b'(\lambda)
- \lambda g'(b(\lambda)) \frac{dx(T, b(\lambda), \lambda)}{db} b'(\lambda) - g(b(\lambda)) \frac{dx(T, b(\lambda), \lambda)}{db} b'(\lambda)
- \lambda g'(b(\lambda)) \frac{dx(T, b(\lambda), \lambda)}{db} b'(\lambda) - \lambda g(b(\lambda)) \frac{d^2 x(T, b(\lambda), \lambda)}{db d\lambda} b'(\lambda).
\]
Using the first-order condition (18), this condition can be simplified to:

\[
\frac{df(b(\lambda), \lambda)}{d\lambda} = \lambda g''(b(\lambda)) [x(T, b(\lambda), \lambda) - x(0)] b'(\lambda) - 2 \lambda g'(b(\lambda)) \frac{dx(T, b(\lambda), \lambda)}{db} b'(\lambda) - \lambda g(b(\lambda)) \frac{d^2x(T, b(\lambda), \lambda)}{dbd\lambda}.
\]

By collecting the terms and setting \( f(b(\lambda), \lambda) = 0 \), we get:

\[
\lambda b'(\lambda) \left( -g''(b(\lambda)) [x(T, b(\lambda), \lambda) - x(0)] - 2 g'(b(\lambda)) \frac{dx(T, b(\lambda), \lambda)}{db} \right) - \lambda g(b(\lambda)) \frac{d^2x(T, b(\lambda), \lambda)}{dbd\lambda} = 0. \tag{20}
\]

The fact that \( b'(\lambda) \leq 0 \) now follows by contradiction. To see this, suppose that \( b'(\lambda) > 0 \). In this case, all terms in equation (20) are positive so they cannot sum to zero. This follows by our assumptions and previous insights, e.g., that \( g'(\cdot) \geq 0, g''(\cdot) \leq 0, g(\cdot) > 0, \lambda > 0, Q(\cdot) > 0, \) and \( dx(\cdot)/db < 0 \).

**Changing the Capacity Constraint \( c \)**

It is next demonstrated that an increase in the capacity from \( c \) to \( c' \) strictly increases the welfare.\(^7\)

Let \( b(c) \) be the maximizer of the welfare function (10) for capacity \( c \). This gives welfare \( W(b(c)) \) and we thus need to show that \( W(b(c')) > W(b(c)) \). When the capacity increases from \( c \) to \( c' \), it follows from equation (7) that the health function \( \int_0^T h(t | b) dt \):

(a) strictly increases if the health-care capacity is exceeded at \( b(c) \) and,

(b) is unaffected if the health-care capacity not is exceeded at \( b(c) \).

Thus, the part of the welfare function that is related to health, i.e., \( (1 - \lambda) \int_0^T h(t | b) dt \), is weakly increasing in \( c \). The production function is independent of \( c \) and therefore, without changing \( b(c) \), the value from the production part in the welfare function, i.e., \( \lambda \int_0^T (1 - g(b(\cdot) \cdot dt) \), is unaffected. Therefore, welfare is weakly higher at \( b(c) \) evaluated at capacity constraint \( c' \). Further, note that \( W(b(c)) \geq W(b) \) for all feasible \( b \). Hence, \( W(b(c')) \geq W(b(c)) \) whenever \( c' > c \).

It remains to show that the latter inequality is strict. But this follows directly by the above arguments and conclusion (a) if the health-care capacity is exceeded at \( b(c) \). The conclusion also follows in the case when the health-care capacity not is exceeded at \( b(c) \) since it then, by construction, is possible to reduce \( b(c) \) without affecting health or production. This will increase welfare by the assumption in Proposition 3 that \( g'(b) > 0 \).

\(^7\)Note also that it is usually costly to increase the health-care capacity, but we are not analyzing that trade-off.