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Appendix and Supplemental material not intended for publication-Round 2

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The appendix for proof of propositions is contained.

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Appendix

A Proofs

Proof of Lemma 1. [2] The consumption level in contract $\bar{\tau}_S$ is determined by equation (7) as

$$\frac{u(\bar{\tau}_S)}{1-\beta} = V^{aut}(\bar{y}_S, 0) = u(c_0^{aut*}) + E\left[\sum_{t=1}^{\infty} \beta^t u(c_t^{aut*})\right]. \tag{10}$$

This can be written as

$$u(\bar{\tau}_S) = (1-\beta)u(c_0^{aut*}) + \sum_{t=1}^{\infty} (\beta^t - \beta^{t+1}) E\left[u(c_t^{aut*})\right]. \tag{11}$$

Therefore, $u(\bar{\tau}_S)$ is a convex combination of the sequence $\{E[u(c_t^{aut*})]\}_{t\geq 0}$. To prove $\bar{\tau}_S > c_0^{aut*}$, it is sufficient to show that

$$\text{for all } t,\ u(c_0^{aut*}) \leq E[u(c_t^{aut*})], \text{ and} \tag{12}$$

$$\text{for some } t,\ u(c_0^{aut*}) < E[u(c_t^{aut*})]. \tag{13}$$

Optimal saving and consumption in autarky necessarily satisfy, for all t,

$$u'(c_t^{aut*}) = \beta(1+r)E_t[u'(c_{t+1}^{aut*})] \quad \text{if} \quad a_t^{aut*} > 0, \text{ and} \tag{14}$$

$$u'(c_t^{aut*}) \geq \beta(1+r)E_t[u'(c_{t+1}^{aut*})] \quad \text{if} \quad a_t^{aut*} = 0. \tag{15}$$

Define a function f by $f = u' \circ u^{-1}$. Then, f^{-1} is decreasing and strictly concave by Assumption 2,[3] and

$$f(u(c_t^{aut*})) = E_t[f(u(c_{t+1}^{aut*}))] \quad \text{if} \quad a_t^{aut*} > 0, \text{ and} \tag{16}$$

$$f(u(c_t^{aut*})) > E_t[f(u(c_{t+1}^{aut*}))] \quad \text{if} \quad a_t^{aut*} = 0. \tag{17}$$

Since the optimal consumption cannot be deterministic after any history, by Jensen's Inequality,

[2] A comment by an anonymous referee helped me simplify the proof.

[3] If the utility function is CARA, then f is linear. In this case, the strict inequality (13) can be obtained by showing that the borrowing constraint binds with positive probability.

we have

$$\text{for all } t,\ u(c_t^{aut*}) < E_t[u(c_{t+1}^{aut*})], \tag{18}$$

which implies (12) and (13). As discussed above, they are sufficient for the result. □

Proof of Proposition 1. Until observing $y_t = \bar{y}_S$, the two strategies yield the same consumption stream. Therefore, it is sufficient to compare the values at the histories y^t in which $y_t = \bar{y}_S$ appears for the first time: $z(y^{t-1}) < \bar{y}_S$ and $z(y^t) = \bar{y}_S$.

First, the value from staying forever without any saving at these histories is given by

$$\frac{u(\bar{\tau}_S)}{1-\beta}. \tag{19}$$

For later use, I transform it as follows:

$$\frac{u(\bar{\tau}_S)}{1-\beta} = u(\bar{\tau}_S) + \beta(1-\Pi_S)\frac{u(\bar{\tau}_S)}{1-\beta} + \beta\Pi_S\frac{u(\bar{\tau}_S)}{1-\beta} \tag{20}$$

$$= u(\bar{\tau}_S) + \beta(1-\Pi_S)\frac{u(\bar{\tau}_S)}{1-\beta} + \beta\Pi_S V^{aut}(\bar{y}_S, 0). \tag{21}$$

The value from strategy $A(\tilde{a})$ is given by

$$u(\bar{\tau}_S - \tilde{a}) + \beta\left[(1-\Pi_S)\frac{u(\bar{\tau}_S + r\tilde{a})}{1-\beta} + \Pi_S V^{aut}(\bar{y}_S, \tilde{a})\right], \tag{22}$$

where $\frac{u(\bar{\tau}_S + r\tilde{a})}{1-\beta}$ is the value of staying forever with initial saving $\tilde{a}$. When staying, he optimally consumes the transfer $\bar{\tau}_S$ and the interest on savings $r\tilde{a}$ and saves $\tilde{a}$ in each period.

The difference between the two values divided by $\tilde{a}$ is

$$-\frac{u(\bar{\tau}_S) - u(\bar{\tau}_S - \tilde{a})}{\tilde{a}} + \beta\frac{1-\Pi_S}{1-\beta}\frac{u(\bar{\tau}_S + r\tilde{a}) - u(\bar{\tau}_S)}{\tilde{a}}$$
$$+\beta\Pi_S\frac{V^{aut}(\bar{y}_S, \tilde{a}) - V^{aut}(\bar{y}_S, 0)}{\tilde{a}},$$

which as $\tilde{a} \to 0$ approaches

$$\begin{aligned} & -u'(\bar{\tau}_S) + \beta \frac{1-\Pi_S}{1-\beta} r u'(\bar{\tau}_S) + \beta \Pi_S (1+r) \frac{dV^{aut}}{da}(\bar{y}_S, 0) & (23) \\ = & -u'(\bar{\tau}_S) + \beta \frac{1-\Pi_S}{1-\beta} r u'(\bar{\tau}_S) + \beta \Pi_S (1+r) u'(c_0^{aut*}) & (24) \\ = & \Pi_S [u'(c_0^{aut*}) - u'(\bar{\tau}_S)] & (25) \\ > & 0. & (26) \end{aligned}$$

The first equality follows from the envelope condition in the autarky problem. To show the second equality, I used $\beta(1+r) = 1$ in Assumption 1 twice. The last inequality follows from lemma 1.

This limit result indicates that there exists some $\tilde{a} > 0$ such that strategy $A(\tilde{a})$ is better than staying forever without saving. □

B The case $v > E[V^{aut}(y_0, 0)]$

The main result is shown under the condition $v = E[V^{aut}(y_0, 0)]$ for simplicity. In the case of $v > E[V^{aut}(y_0, 0)] = \bar{w}_1$, the efficient contract in the environment without saving has almost the same structure, as explained in Ljungqvist and Sargent (2012).

Given such v, it holds that $\bar{w}_S < v$ or that there exists k such that $\bar{w}_k < v \leq \bar{w}_{k+1}$. In the first case, the efficient contract is simply giving a constant transfer $\tilde{\tau}$ such that $u(\tilde{\tau})/(1-\beta) = v$. If v is sufficiently close to $\bar{w}_S$, then Proposition 1 still holds. When v is high and not close to $\bar{w}_S$, staying in the contract is better than returning to autarky, and the result does not hold. In the second case, the efficient contract gives $\tilde{\tau}$ such that

$$v = \left(\sum_{s=1}^{k-1} \Pi_s \right) [u(\tilde{\tau}) + \beta v] + \sum_{s=k}^{S} \Pi_s [u(\bar{\tau}_s) + \beta \bar{w}_s] \quad (27)$$

as long as the maximum endowment in the past history is less than or equal to $\bar{y}_{k-1}$. When the maximum endowment in the past history is $\bar{y}_j$ for $j \geq k$, the efficient contract gives the transfer $\bar{\tau}_j$. As a result, the failure of incentive compatibility can be shown for this case by the same proof.