E C O N O M I C S   B U L L E T I N

Calibration results for rank−dependent expected utility

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Abstract

If its utility function is everywhere increasing and concave, rank−dependent expected utility shares a troubling property with expected utility −− aversion to the same moderate−stakes risk at every wealth level implies an extreme aversion to large−stakes risks. In fact, the problem may be even worse for rank−dependent expected utility, since the moderate−stakes risk need not be actuarially fair.

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1. Introduction

In addition to the long list of experimental studies questioning its validity, there are now theoretical reasons why the expected utility model might be inappropriate. A property of expected utility for small gambles has long been known – for any gamble $\tilde{x}$ and for sufficiently small $t > 0$, an expected utility maximizer with a smooth utility function prefers the gamble $t\tilde{x}$ to its expected value if $\tilde{x}$ has positive expected value. So, expected utility maximizers are risk neutral for lotteries with sufficiently small payoffs. Rabin (2000) discusses the other end of the spectrum. He shows that for any increasing, concave utility function, an expected utility maximizer who is averse to the same moderate risk at every wealth level must necessarily be extremely averse to large risks. In perhaps his most striking example, somebody who turns down a 50:50 lose \$100/gain \$125 gamble at every wealth level must turn down *any* gamble with a 50% chance of losing \$600, no matter how large the gain is. Putting his result another way, an expected utility maximizer who is averse to *any* single actuarially favorable gamble at every wealth level must be extremely (even ridiculously) risk averse over large gambles. This argument provides a severe criticism of standard expected utility models.

The purpose of this note is to explore whether Rabin's result holds when expected utility is weakened. In particular, I examine the result in the context of rank-dependent expected utility (Quiggin, 1982, 1993). There are several good reasons for doing so. First, among nonexpected utility models, rank-dependent expected utility has received the most recent attention and has been used in the most applications. Second, rank-dependent expected utility exhibits first-order risk aversion, i.e. the risk premium for the noise variable $t\tilde{\varepsilon}$ is proportional to t at $t = 0$, whereas expected utility exhibits second-order risk aversion, where the risk premium is proportional to t^2 at $t = 0$ (see Segal and Spivak, 1990). First-order risk aversion has been used to explain a number of empirical anomalies, such as the purchase of full insurance with a loading factor (Segal and Spivak, 1990), the equity-premium puzzle (Epstein and Zin, 1990), and tax compliance (Bernasconi, 1998). Perhaps second-order risk aversion is the driving force behind Rabin's result. The results of the rank-dependent expected utility analysis are fairly simple – Rabin's negative result still holds. Furthermore, experimental evidence and theoretical work on the shape of the probability weighting function suggest that the rank-dependent result is even worse than the expected utility one. With rank-dependent expected utility and the most common current weighting functions, aversion to a single, appropriately chosen, *actuarially unfavorable* gamble at every wealth level can lead to extreme levels of risk aversion over large gambles.

The results have two implications for Rabin's criticism of expected utility. First, it is unrelated to the properties of approximate risk neutrality and second-order risk aversion, since for rank-dependent expected utility approximate risk neutrality fails but Rabin's criticism is maintained. Second, his criticism extends to an important generalization of expected utility theory.

Section 2 briefly reviews Rabin's result for expected utility preferences. Section 3 explores the same result for rank-dependent expected utility preferences.

2. Expected Utility

An expected utility maximizer has a preference function $V(F)$ over probability distributions taking the form $V(F) = \int U(w + x)dF(x)$, where U is the utility function defined over wealth levels. Rabin's calibration result uses the following construction:

$$m(k)=\begin{cases}\dfrac{\ln\left[1-\left(1-\frac{l}{g}\right)2\sum\limits_{i=1}^{k}\left(\frac{g}{l}\right)^{i}\right]}{\ln\frac{l}{g}}-1 & \text{if } 1-\left(1-\frac{l}{g}\right)2\sum\limits_{i=1}^{k}\left(\frac{g}{l}\right)^{i}>0\\ \infty & \quad\;\; 1-\left(1-\frac{l}{g}\right)2\sum\limits_{i=1}^{k}\left(\frac{g}{l}\right)^{i}\leq 0\end{cases}\qquad(1)$$

Rabin's Expected Utility Calibration Theorem. *Let U be twice differentiable with $U'(w)>0$, $U''(w)<0$ for all w. If there exists $g>l>0$ such that, for all w, an expected utility maximizer with utility function U prefers not to take a 50:50 chance of losing l and winning g, then she also prefers not to take a 50:50 chance of losing kl and winning mg for any positive integer k and any $m<m(k)$ as defined in expression (1).*

Rabin's result takes the following form: If an individual turns down 50:50 lose l/gain g bets at every wealth level, then the same individual also turns down a 50:50 lose kl/gain mg bet where k is a positive integer and m is bounded by $m(k)$ defined above. Some values of l, g, and k yield an infinite value for $m(k)$, so that the individual will turn down *any* bet involving a 50% chance of losing kl, independent of the amount of the potential gain. Furthermore, for any $g>l>0$, there exists an integer $K_\infty<0$ such that for $k\geq K_\infty$, $m(k)=\infty$. This means that if the individual turns down any 50:50 lose l/gain g gamble, there is a large enough integer K_∞ such that she turns down *any* gamble with a 50% chance of losing $K_\infty\cdot l$ or more. The value K_∞ is the smallest integer k for which

$$1-\left(1-\frac{l}{g}\right)2\sum_{i=1}^{k}\left(\frac{g}{l}\right)^{i}\leq 0,$$

which depends only on the ratio g/l. Table 1 gives the values of K_∞ for different values of g/l. As shown by the table, K_∞ can take on some small values, implying that the individual is extremely risk averse. So, for example, if an individual is averse to a 50:50 chance of losing \$100 or gaining \$105 at every wealth level, then she will not take any gamble with a 50% chance of losing \$900, no matter how large the gain. Even more strikingly, being averse to a 50:50 chance of losing \$$l$ or gaining \$$1.5l$ at every wealth level means that she will not take *any* gambles with a 50% chance of losing l. This is a serious criticism of the expected utility hypothesis..

Table 1.

g/l	**K_∞**	**g/l**	**K_∞**
1.01	41	1.07	6
1.02	21	1.09	5
1.03	14	1.11	4
1.04	11	1.15	3
1.05	9	1.23	2
1.06	7	1.5	1

3. Rank-dependent expected utility

Rank-dependent expected utility preferences differ from expected utility preferences in that the probability distribution is transformed before expected utility is calculated. The function π is defined to be a probability transformation function if it is strictly increasing with $\pi(0) = 0$ and $\pi(1) = 1$. The preference function takes the form $V(F) = \int U(w)d\pi(F(w))$ where U is a utility function, as before, and π is a probability transformation function. Risk attitudes are governed by both U and π – if both are concave the individual is risk averse.

When there are only two outcomes, as there are in Rabin's setting, the rank-dependent expected utility preference function takes a very simple form. Letting p_l be the probability of the loss, the preference function takes the form $\pi(p_l)U(w - l) + [1 - \pi(p_l)]U(w + g)$. So, the preference function differs from the expected utility preference function in that the expectation is taken using $\pi(p_l)$ as the probability of the loss instead of p_l.

Proposition 1. *Let U be twice differentiable with $U'(w) > 0$, $U''(w) < 0$ for all w. Let π be a probability transformation function, and let $\bar{p} = \pi^{-1}\left(\frac{1}{2}\right)$. If there exists $g > l > 0$ such that, for all w, a rank-dependent expected utility maximizer with utility function U and transformation function π prefers not to take a $\bar{p} : (1-\bar{p})$ chance of losing l and winning g, then she also prefers not to take a $\bar{p} : (1 - \bar{p})$ chance of losing kl and winning mg for any positive integer k and any $m < m(k)$ as defined in expression (1).*

Proof. If the individual prefers not to take the gamble, it must be the case that

$$\pi(\bar{p})U(w - l) + [1 - \pi(\bar{p})]U(w + g) < U(w).$$

Since $\pi(\bar{p}) = \frac{1}{2}$, this implies that $\frac{1}{2}U(w-l)+\frac{1}{2}U(w+g) < U(w)$. By Rabin's EU calibration theorem, $\frac{1}{2}U(w - kl) + \frac{1}{2}U(w + mg) < U(w)$ for any positive integer k and any $m < m(k)$ given by (1). Thus,

$$\pi(\bar{p})U(w - kl) + [1 - \pi(\bar{p})]U(w + mg) < U(w)$$

for any positive integer k and any $m < m(k)$. ■

The only difference between Rabin's expected utility calibration theorem and its rank-dependent counterpart is the probability of the loss. The conditions on the utility function are identical. So, given an increasing concave utility function U and a probability transformation function π, if g, l, and k satisfy the conditions of Rabin's EU theorem, they also satisfy the conditions of Proposition 1, but using $\bar{p}$ as the probability of the loss instead of $\frac{1}{2}$. This is unsurprising because $\bar{p}$ is simply the solution to $\pi(p) = \frac{1}{2}$, so that the transformed probability distribution coincides with the 50:50 one used in Rabin's EU calibration theorem. Consequently, the problem posed for expected utility by Rabin's analysis also pertains to rank-dependent expected utility. Aversion to an appropriate moderate-stakes risk at every wealth level implies extreme aversion to risks with large stakes.

Even though the expected utility calibration result and its rank-dependent counterpart are very similar, their interpretations hold significant differences. Most experimental studies of the probability transformation function π, including Tversky and Kahneman (1992), Camerer and Ho (1994), Wu and Gonzales (1996), Fox, Rogers, and Tversky (1996), and Gonzales and Wu (1999), along with Prelec's (1998) axiomatic characterization of the transformation function, find that $\pi^{-1}(\frac{1}{2}) > \frac{1}{2}$. For Proposition 1, this means that the $\bar{p} : (1-\bar{p})$ chance of losing l or winning g may be actuarially unfavorable. So, for example, if $\pi^{-1}(\frac{1}{2}) = 0.6$, if the individual dislikes 60:40 lose $100/win $110 gambles at every wealth level, by Proposition 1 and Table 1, that individual will not take any gamble with a 60% chance of losing $500. Disliking the 60:40 lose $100/win $110 gamble does not require risk aversion, but instead requires a sufficiently small amount of risk preference. Disliking any gamble with a 60% chance of losing $500, though, requires an extreme amount of risk aversion. Consequently, the negative calibration result for rank-dependent expected utility using the now-standard specifications of the probability transformation function is even more striking than its expected utility counterpart.

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