E C O N O M I C S B U L L E T I N

Existence of subgame perfect equilibrium with public randomization: A short proof

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Abstract

Consider a multi−stage game where each player has a compact choice set and payoffs are continuous in all such choices. Harris, Reny and Robson (1995) prove existence of a subgame perfect equilibrium as long as a public correlation device is added to each stage. They achieve this by showing that the subgame perfect equilibium path correspondence is upper hemicontinuous. The present paper gives a short proof of existence that focuses on equilibrium payoffs rather than paths.

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1. Introduction

Games with infinite choice sets are natural in many economic applications. Nevertheless, existence of a suitable equilibrium is often a non-trivial issue. For example, consider a game with a finite number of stages, in each of which players move simultaneously with complete information on previous choices. Harris, Reny and Robson (1995) give an example of such a game in which there is no subgame perfect equilibrium (SPE) despite the compactness of choice sets at each stage and the continuity of payoffs. On the other hand, existence of an SPE is guaranteed if at each stage a publicly observed signal is added which permits individuals at that stage to correlate over the set of equilibria in the continuation game.

In fact, Harris, Reny and Robson establish this result by showing that the SPE *path* correspondence is upper hemicontinuous. The present paper shows that, by focussing on equilibrium *payoffs* rather than paths, a substantially shorter proof of existence per se is available.

2. The Class of Games

A *continuous stage game with public randomization* is comprised of N players playing simultaneously in each of T stages. Within each stage after the first, Nature first chooses a public signal drawn from a uniform distribution on $[0,1]$, independently of all previous choices and signals. All players are informed of this signal as well as of all choices made in all previous stages. Players then make their simultaneous choices at the given stage. Our notation is as follows.

Choices/strategies. The set of choices available to player i at every stage is contained in a fixed compact metric space of choices, C_i. Let $C \equiv \times_{i=1}^{N} C_i$. A typical choice of player i at stage t is denoted c_i^t, and the vector of stage t choices $(c_1^t, ..., c_N^t)$ of the N players is denoted by c^t. Nature's stage t signal is denoted by $s^t \in [0,1]$, for $t \geq 2$.

Player i's compact metric space of possible stage 1 choices is denoted $C_i^1 \subseteq C_i$ and $C^1 \equiv \times_{i=1}^{N} C_i^1$. For stages $t = 2, ..., T$, a history prior to t, denoted h^{t-1}, is a vector of the form $h^{t-1} = (c^1, s^2, c^2, ..., s^{t-1}, c^{t-1})$. The set of all such possible histories prior to t is denoted by H^{t-1}. For each such stage t, let $\mathcal{C}_i^t(h^{t-1})$ denote the set of possible choices available to player i given the history h^{t-1}. Thus given h^{t-1}, the joint choice vector c^t satisfies $c^t \in \mathcal{C}_1^t(h^{t-1}) \times ... \times \mathcal{C}_N^t(h^{t-1}) \equiv \mathcal{C}^t(h^{t-1}) \subseteq C$. It is assumed that $\mathcal{C}^t : H^{t-1} \twoheadrightarrow C$ is a continuous correspondence which is independent of $s^2, ..., s^{t-1}$. For all $t \geq 2$, it follows that H^t is a compact metric space, as are the sets $\mathcal{C}^t(h^{t-1})$, for each h^{t-1}.

A *strategy* for player i is a $T-$tuple $(\alpha_i^1, \alpha_i^2, ..., \alpha_i^T)$ such that (i) $\alpha_i^1 \in \Delta(C_i^1)$,[1] (ii) for all $t \geq 2$, $\alpha_i^t : H^{t-1} \times [0,1] \to \Delta(C_i)$ is Borel measurable, and (iii) for all $t \geq 2$ and all $(h^{t-1}, s^t) \in H^{t-1} \times [0,1]$, $\alpha_i^t(\mathcal{C}_i^t(h^{t-1})|h^{t-1}, s^t) = 1$. We refer to α_i^t as player i's stage t strategy.

Payoffs. $U_i : H^T \to \mathbb{R}_{++}$ is continuous and independent of $(s^2, ..., s^T)$ for every $i = 1, ..., N$.[2]

Our objective is to prove the following.

Theorem 1. Every continuous stage game with public randomization possesses a subgame perfect equilibrium.

3. Existence of Subgame Perfect Equilibrium

We first employ a backward induction program to generate a suitable payoff correspondence from which to select an SPE payoff function. However, this alone does not guarantee that the underlying strategies are measurable functions of histories, and so also fails to ensure that a player can calculate the payoff consequences of an arbitrary deviation. The requisite measurablility is ensured through an additional forward induction program.

The backward and forward programs. We first sketch the backward induction program, which selects a set of continuation payoffs for each stage, beginning with stage T. Fixing a stage t, suppose this program has so far generated, for each history h prior to t, for each stage t signal of nature s, and for each stage t choice vector $c \in C^t(h)$, a set of stage t continuation payoff vectors $P(h, s, c)$. Suppose further that this correspondence P is non-empty-valued, convex-valued and upper hemicontinuous (u.h.c.). (These conditions on P hold for $t = T$ since $P(h, s, c)$ is then $\{u(h, s, c)\}$.) For each such h and s, choose $p \in \mathcal{S}(P(h, s, \cdot))$,[3] and consider the set of NE in the game with pure strategies $\mathcal{C}^t(h)$ and payoff function $p : \mathcal{C}^t(h) \to \mathbb{R}_{++}^N$. Let $Q(h, s)$ denote the set of NE payoff vectors obtained from all such measurable selections from $P(h, s, \cdot)$. Because Nature chooses a public signal uniformly from $[0,1]$ at the beginning of each stage, the expected payoff vector given h, just before Nature chooses the signal, belongs to the set $\int_0^1 Q(h, s)ds \equiv \bar{P}(h)$. The backward inductive step is to show that $\bar{P}(h)$ is also

[1] $\Delta(A)$ is the set of Borel probability measures on a compact metric space A, so that $\Delta(A)$ is itself a compact metric space with the topology of weak convergence.

[2] Assuming payoffs are strictly positive facilitates, for instance, the proof of Lemma 6 in the Appendix.

[3] If $G(\cdot)$ is any correspondence, then $\mathcal{S}(G(\cdot))$ is the set of measurable selections from $G(\cdot)$.

non-empty-valued, convex-valued and u.h.c. and so can play in stage $t-1$ the role played by P in stage t.

Suppose the above backward induction program has been completed. *Measurable* strategies underlying these payoffs are then obtained by a *forward* induction program beginning in stage 1 and ending in stage T, sketched as follows. Suppose, at the beginning of stage t, that this program has constructed for each player i, measurable stage 1 through stage $t-1$ strategies that form an SPE in the $t-1$ stage truncation of the game, and that the payoff vector from each history $h \in H^{t-1}$ is $\bar{p}(h)$, for some $\bar{p} \in S(\bar{P}(\cdot))$, where $\bar{P}(\cdot)$ is the correspondence derived in the backward program for the $(t-1)$st stage game. The forward inductive step is to show that there exists a measurable payoff selection $p : H^t \rightarrow R^N_{++}$ from $P(\cdot)$, the payoff correspondence from the backward program for the tth stage, such that $\bar{p}$ can be induced from p and *measurable* stage t strategies. That is, that there exists a measurable function $\alpha : H^{t-1} \times [0,1] \rightarrow \times_{i=1}^N \Delta(C_i)$ such that for all $h \in H^{t-1}$ and $s \in [0,1]$, $\bar{p}(h) = \int_0^1 \int_C p(h,s,c) d\alpha(c|h,s) ds$ and $\alpha(\cdot|h,s)$ is an NE in the subgame (h,s) with the payoff selection $p(h,s,\cdot)$. This inductive step yields measurable stage 1 through stage t strategies constituting an SPE in the t stage truncation of the original game with payoff vector given by p.

To prove Theorem 1, it suffices then to show that a typical *single* step in both the backward and forward induction programs can be performed.

Proof of Theorem 1. For simplicity, we avoid all explicit reference to the particular stage game under consideration. The notation is otherwise as in the above description of the backward and forward programs. Let $F \equiv \{(h,s,c) \in H \times [0,1] \times C : c \in \mathcal{C}(h)\}$, where $\mathcal{C}(h) \equiv \times_{i=1}^N \mathcal{C}_i(h) \subseteq C$. Thus F is the compact set of feasible history-signal-choice triples up to the end of the stage game in question, and the set of available choices given h, $\mathcal{C}(h)$, varies continuously with $h \in H$. Let $P : F \twoheadrightarrow \mathbb{R}^N_{++}$ and $\bar{P} : H \twoheadrightarrow \mathbb{R}^{\mathbf{N}}_{++}$ be the payoff correspondences defined in the description of the backward induction program.

Proposition 2. (Backward Inductive Step) Suppose that P is nonempty-valued, convex-valued, u.h.c., and independent of all signals of Nature. Then $\bar{P}$ has these properties as well.

Proof. Simon and Zame (1990) implies that for every $(h,s) \in H \times [0,1]$ there exists a measurable selection from $P(h,s,\cdot)$ and an associated NE $\alpha \in \Delta(\mathcal{C}(h))$. Let $Q(h,s) \subseteq \mathbb{R}^{\mathbf{N}}_{++}$ then denote the nonempty set of all Nash equilibrium payoff vectors that result from some such selection. Note that since P is independent of s, so is Q. Thus we write $Q(h)$ rather than $Q(h,s)$. Since by definition, $\bar{P}(h) = \int_0^1 Q(h) ds$, it follows that $\bar{P}(h) = co\ Q(h)$, the convex hull of $Q(h)$, for all $h \in H$.

(See, for instance, Hildenbrand, 1974, p. 62.) Hence $\bar{P}$ is nonempty-valued and convex-valued. Finally $\bar{P}$ must be shown to be u.h.c. This follows from the u.h.c. of Q established in the Remark following the proof of Lemma 4. ■

Proposition 3. (Forward Inductive Step) Suppose that $P : F \twoheadrightarrow \mathbb{R}^N_{++}$ is nonempty-valued, convex-valued and u.h.c. and that $P(h, s, c)$ is independent of s for every $h \in H$ and $c \in \mathcal{C}(h)$. Suppose that $\bar{P} : H \twoheadrightarrow \mathbb{R}^N_{++}$ is nonempty-valued, convex-valued and u.h.c. Then for any $\bar{p} \in \mathcal{S}(\bar{P}(\cdot))$ there exists $p \in \mathcal{S}(P(\cdot))$ and a *measurable* map $\alpha : H \times [0,1] \rightarrow \times_{i=1}^N \Delta(C_i)$ such that for all $h \in H$ and $s \in [0,1]$,

(i) $\alpha(\cdot|h,s) \in \times_{i=1}^N \Delta(\mathcal{C}_i(h))$

(ii) $\bar{p}(h) = \int_0^1 \int_C p(h,s,c) d\alpha(c|h,s) ds$,

(iii) $\alpha(\cdot|h,s)$ is an NE in the subgame (h,s) given the payoff selection $p(h,s,\cdot)$.

Proof. We shall demonstrate the existence of measurable strategies, α, that are, in addition, step functions of the current signal by Nature. Further, since payoffs belong to R^N_{++}, Carathéodory's Theorem will imply that these step functions require at most $N+1$ distinct values.

Step 1. Fix an arbitrary $\bar{s} \in [0,1]$. Define the correspondence $\Psi : H \twoheadrightarrow \mathbb{R}^{\mathbf{N}}_{++} \times ([0,1] \times \Delta(C) \times M(C))^{N+1}$ as follows:[4] $(v, (s^k, \alpha^k, \mu^k)_{k=0}^N) \in \Psi(h)$ iff, for $k = 0, ..., N$,

(a) $0 \equiv s^0 \leq s^1 \leq ... \leq s^N \leq s^{N+1} \equiv 1$,

(b) $v = \sum_{k=0}^N (s^{k+1} - s^k) \int_C \tilde{p}^k(h,c) d\alpha^k$, for some $\tilde{p}^k(h,\cdot) \in \mathcal{S}(P(h,\bar{s},\cdot))$, for all $k = 0, ..., N$,

(c) $\alpha^k \in \times_{i=1}^N \Delta(\mathcal{C}_i(h))$ is a NE in subgame $(h,\bar{s})$ with payoffs $\tilde{p}^k(h,\cdot)$,

(d) $\mu^k = \tilde{p}^k \alpha^k$.

The notation in (d) above means that $\mu^k(A) = \int_A \tilde{p}^k(h,c) d\alpha^k(c)$, for all $A \in \mathcal{B}(\mathcal{C}(h))$. (If D is any metric space, $\mathcal{B}(D)$ denotes its Borel σ-algebra.) Since $\bar{P}$ is nonempty-valued, so too is Ψ. Lemma 4 in the Appendix defines a related correspondence $\Phi : H \twoheadrightarrow \mathbb{R}^{\mathbf{N}}_{++} \times \Delta(C) \times M(C)$ and shows that Φ is u.h.c. That Ψ is u.h.c. then follows.

Step 2. Let $g : H \times \mathbb{R}^{\mathbf{N}}_{++} \times ([0,1] \times \Delta(C) \times M(C))^{N+1} \rightarrow \mathbb{R}^{\mathbf{N}}_{++}$ be the projection map defined by $g(h, v, (s^k, \alpha^k, \mu^k)_{k=0}^N) \equiv v$. Since by Step 1, Ψ is nonempty-valued and u.h.c., it is compact-valued and measurable. Also, by definition of $\bar{P}$ and Ψ and by Carathéodory's Theorem, $\bar{p}(h) \in g(\{h\} \times \Psi(h))$ for all $h \in H$. Consequently, by Lemma 5 in the Appendix, there exists $\psi \in \mathcal{S}(\Psi(\cdot))$ satisfying $\bar{p}(h) = g(h, \psi(h))$ for all h. That is, there exist, for $k = 0, 1, ..., N$, *measurable* maps $s^k : H \rightarrow [0,1]$, where $0 \equiv s^0(h) \leq s^1(h) \leq ... \leq s^N(h) \leq s^{N+1}(h) \equiv 1$, *measurable*

[4]Note that $\times_{i=1}^N \Delta(C_i(h)) \subseteq \Delta(C)$. If $M(C)$ is the set of positive bounded measures on any compact metric space C, then $\times_{i=1}^N M(C_i) \subseteq M(C)$.

maps $\alpha^k : H \to \times_{i=1}^N \Delta(C_i)$, where $\alpha^k(\cdot|h) \in \times_{i=1}^N \Delta(\mathcal{C}_i(h))$, and *measurable* maps $\mu^k : H \to M(C)$ satisfying, for all $h \in H$:
1. $\bar{p}(h) = \sum_{k=0}^{N} (s^{k+1}(h) - s^k(h)) \int_C \tilde{p}^k(h,c) d\alpha^k(c|h)$, for some $\tilde{p}^k(h,\cdot) \in \mathcal{S}(P(h,\bar{s},\cdot))$ for all $h \in H$,
2. $\alpha^k(\cdot|h)$ is an NE in the subgame $(h,\bar{s})$ given the payoff selection $\tilde{p}^k(h,\cdot)$,
3. $\mu^k(\cdot|h) = \tilde{p}^k(h,\cdot)\alpha^k(\cdot|h)$.

Step 3. Although $\tilde{p}^k(\cdot)$ need not be $\mathcal{B}(H \times C)$-measurable, it can be replaced by some $p^k(\cdot)$ which is measurable. In the light of "1" and "3" from Step 2, Lemma 6 in the Appendix implies that for each k, there exists $\hat{p}^k \in \mathcal{S}(P(\cdot,\bar{s},\cdot))$ such that $\hat{p}^k(h,c) = \tilde{p}^k(h,c)$ for $\alpha^k(\cdot|h)-$a.e. $c \in \mathcal{C}(h)$, for $k = 0,...,N$ and all $h \in H$.

Consider now all choices for player i which, in the subgame determined by $(h,\bar{s})$, are better for i under $\hat{p}_i^k$ than playing according to α_i^k under $\tilde{p}_i^k$:

$$B_i^k(h) \equiv \{\bar{c}_i \in \mathcal{C}_i(h) : \int_{C_{-i}} \hat{p}_i^k(h,\bar{c}_i,c_{-i}) d\alpha_{-i}^k(c_{-i}|h) > \int_C \tilde{p}_i^k(h,c) d\alpha^k(c|h)\}.$$

Clearly $B_i^k(h)$ is measurable, for each $h \in H$. Further, given "2" of Step 2 and the $\alpha^k(\cdot|h)-$a.e. equality of $\hat{p}^k(h,\cdot)$ and $\tilde{p}^k(h,\cdot)$, it follows that $\alpha_i^k(B_i^k(h)|h) = 0$. Let $\check{p}^i \in \mathcal{S}(P(\cdot,\bar{s},\cdot))$ be a *vector* payoff in $\mathbb{R}_{++}^N$ minimizing player i's payoff and define

$$p^k(h,c) = \begin{cases} \check{p}^i(h,c), \text{ if } (h,c_i) \in Gr(B_i^k), \text{ but } (h,c_j) \notin Gr(B_j^k), \ \forall j \neq i \\ \hat{p}^k(h,c), \text{ otherwise.} \end{cases}$$

It can be shown that $Gr(B_l^k)$, the graph of B_l^k, is a measurable subset of $H \times C_l$, for each player l.[5] It follows that p^k is $\mathcal{B}(H \times C)$-measurable. Hence, $p^k \in \mathcal{S}(P(\cdot,\bar{s},\cdot))$. Moreover, since by construction, $p^k(h,\cdot)$ agrees $\alpha^k(\cdot|h)-$a.e. with $\tilde{p}^k(h,\cdot)$, and i's expected payoff under $p^k(h,\cdot)$ from any available choice cannot exceed that under $\tilde{p}^k(h,\cdot)$, $\alpha^k(\cdot|h)$ is an NE in the subgame $(h,\bar{s})$ given the payoff selection $p^k(h,\cdot)$, for all $k = 0,1,...,N$, and $h \in H$.

Step 4. Finally, define $p : F \to \mathbb{R}_{++}^N$ by $p(h,s,c) \equiv p^k(h,c)$ if $s^k(h) \leq s < s^{k+1}(h)$; and define $\alpha : H \times [0,1] \to \times_{i=1}^N \Delta(C_i)$ by $\alpha(\cdot|h,s) \equiv \alpha^k(\cdot|h)$ if $s^k(h) \leq s < s^{k+1}(h)$ where for $k = N$, the strict inequalities in both definitions are weak.

Since for every k the functions p^k, α^k, and s^k are Borel measurable, so are the functions p and α. Moreover, $p(h,s,c) \in P(h,s,c)$ for all $(h,s,c) \in F$. This

[5] Dubins and Freedman (1964, 2.2 and 3.1) imply that $\int_C \tilde{p}_i^k(h,c) d\alpha^k(c|h)$ is a Borel measurable function of h. Indeed, it follows that $\int_{C_{-i}} \hat{p}_i^k(h,\bar{c}_i,c_{-i}) d\alpha_{-i}^k(c_{-i}|h)$ is also a Borel measurable function of $(\bar{c}_i,h)$.

is since, by Step 3, $p^k(h,c) \in P(h,\bar{s},c)$, for all $h \in H$ and $c \in \mathcal{C}(h)$, and since $P(h,s,c)$ is independent of s. It follows that $p \in \mathcal{S}(P(\cdot))$.

By "1" and "2" of Step 2, p and α satisfy (i) and (ii). Finally, (iii) follows from the conclusion of Step 3 and the fact that $P(h,s,c)$ is independent of s. ■

4. Appendix

Lemma 4. (Simon and Zame 1990): Suppose that H, S and C_i, $i = 1, 2, ..., N$ are compact metric spaces and that $C \equiv \times_{i=1}^{N} C_i$. Let $P : H \times S \times C \twoheadrightarrow \mathbb{R}^N_{++}$ be a u.h.c., convex-valued, and nonempty-valued payoff correspondence, and $\mathcal{C} : H \twoheadrightarrow C$ be a continuous choice correspondence. It follows that: (i) For every $(h,s) \in H \times S$, there is a payoff selection $p(\cdot) \in \mathcal{S}(P(h,s,\cdot))$ such that the $N-$player game with pure choice sets $\mathcal{C}_i(h) \subseteq C_i$ $i = 1, 2, ..., N$ and payoff function $p(\cdot)$ possesses a mixed strategy Nash equilibrium; (ii) The correspondence $\Phi : H \twoheadrightarrow \mathbb{R}^{\mathbf{N}}_{++} \times \Delta(C) \times M(C)$ defined by $(v, \alpha, \mu) \in \Phi$ iff

(b′) $v = \int_C \tilde{p}(h,c) d\alpha$, for some $\tilde{p}(h,\cdot) \in \mathcal{S}(P(h,\bar{s},\cdot))$,

(c′) $\alpha \in \times_{i=1}^{N} \Delta(\mathcal{C}_i(h)$ is an NE in the subgame $(h,\bar{s})$ with payoffs $\tilde{p}(h,\cdot)$,

(d′) $\mu = \tilde{p}\alpha$,

is u.h.c. (Here, $\bar{s} \in [0,1]$ is fixed but arbitrary.)

Proof. Part (i) follows directly from Simon and Zame (1990) by noting that for every $(h,s) \in H \times S$, the correspondence $P(h,s,\cdot) : \mathcal{C}(h) \twoheadrightarrow \mathbb{R}^N_{++}$ satisfies the conditions of Section 2 of their paper. Moreover, their Theorem proves (ii) as well, reinterpreting the index on their sequence of finite approximations to players' strategy sets as a sequence of h's here. Their requirement that the sequence of strategy sets yields a Hausdorff approximation to the original strategy sets is ensured here by continuity of the choice correspondence $\mathcal{C}(h)$ in h.[6] ■

Remark: It follows from (ii) that $Q(\cdot)$, as in the proof of Proposition 2, and $\Psi(\cdot)$, as in Step 1 of the proof of Proposition 3, are u.h.c. correspondences.

For the remaining two results it is assumed that H and C are compact metric spaces. This assumption is stronger than needed for the results quoted.

Lemma 5. Suppose that $\Psi : H \twoheadrightarrow \mathbb{R}^{\mathbf{N}} \times (\mathbb{R} \times \Delta(C) \times M(C))^{N+1}$ is a compact-valued measurable correspondence **(**Wagner (1977, pp. 862-863))**,** $g : Gr(\Psi) \to \mathbb{R}^{\mathbf{N}}$ is a Carathéodory map **(**Wagner (1977, p. 863)**).** and $\bar{p} : H \to \mathbb{R}^{\mathbf{N}}$ is measurable with $\bar{p}(h) \in g(\{h\} \times \Psi(h))$, for all $h \in H$. Then there exists $\psi \in \mathcal{S}(\Psi)$ such that $\bar{p}(h) = g(h, \psi(h))$ for every $h \in H$.

[6] Continuity of the choice correspondence also ensures that the limit of a sequence of Nash equilibrium distributions is feasible.

Proof. See Wagner (1977, p. 876, Theorem 7.4 (iii).)

Lemma 6. Suppose that $P : H \times C \twoheadrightarrow \mathbb{R}^N$ is nonempty-valued and u.h.c. and the maps $\alpha : H \to \Delta(C)$ and $\mu : H \to \Delta(C)$ are Borel measurable. In addition, suppose $\mu(A|h) = \int_A \tilde{p}(h,c)d\alpha(c|h)$, for all $A \in \mathcal{B}(C)$, where $\tilde{p}(h,\cdot)$ is for every h a $\mathcal{B}(C)-$measurable selection from $P(h,\cdot)$. Then although $\tilde{p}$ need not be $\mathcal{B}(H \times C)-$measurable, there exists a selection $\hat{p}(\cdot)$ from $P(\cdot)$ which is measurable, such that $\mu(A|h) = \int_A \hat{p}(h,c)d\alpha(c|h)$, for all $h \in H$ and all $A \in \mathcal{B}(C)$. Thus $\hat{p}(h,c) = \tilde{p}(h,c)$ for $\alpha(\cdot|h)$-a.e. $c \in C$, for each $h \in H$.

Proof. Since α and μ are Borel measurable functions, $\alpha(A|h)$ and $\mu(A|h)$ are measurable functions of h for every fixed Borel subset A of C. **(**See Dubins and Freedman (1964, 2.2 and 3.1), for example.**)** Consequently, indexed by h, $\alpha(\cdot|h)$ and $\mu(\cdot|h)$, after normalizing $\mu(\cdot|h) \in M(C)$ to be a probability measure, are a measurable family of probability measures. **(** See Dellacherie and Meyer, 1978, p. 20, II.13. Therefore, by Theorem V.58 of Dellacherie and Meyer (1982, p. 52), there exists a $\mathcal{B}(H \times C)-$measurable function $p' : H \times C \to \mathbb{R}^N$ such that $\mu(A|h) = \int_A p'(h,c)d\alpha(c|h)$, for all $h \in H$ and all $A \in \mathcal{B}(C)$. Let

$$E \equiv \{(h,c) : p'(h,c) \in P(h,c)\} = \{(h,c) : [P(h,c) \cap \{p'(h,c)\}] \neq \emptyset\}.$$

The correspondence in square brackets is measurable, being the intersection of two measurable correspondences whose values are closed subsets of $\mathbb{R}^N$. (See Wagner, 1977, p. 863.) Consequently E is a measurable subset of $H \times C$. Define

$$\hat{p}(h,c) = \begin{cases} p'(h,c) \text{ if } (h,c) \in E \\ m(h,c) \text{ otherwise} \end{cases}$$

where $m \in \mathcal{S}(P)$ is arbitrary. It follows that $\hat{p} \in \mathcal{S}(P)$. For every $h \in H$, $p'(h,c) = \tilde{p}(h,c)$, for $\alpha(\cdot|h)-$a.e. $c \in C$. Recalling that $\tilde{p}(h,\cdot) \in \mathcal{S}(P(h,\cdot))$, it follows that $\alpha(\{c : p'(h,c) \in P(h,c)\}|h) = 1$, for each $h \in H$. Hence

$$\mu(A|h) = \int_A p'(h,c)d\alpha(c|h) = \int_A \hat{p}(h,c)d\alpha(c|h),$$

for all $h \in H$ and all $A \in \mathcal{B}(C)$. ∎

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