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Strategy−proofness of social choice functions and non−negative association property with continuous preferences

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Abstract

We consider the relation between strategy−proofness of resolute (single−valued) social choice functions and its property which we call Non−negative association property (NNAP) when individual preferences over infinite number of alternatives are continuous, and the set of alternatives is a metric space. NNAP is a weaker version of Strong positive association property (SPAP) of Muller and Satterthwaite(1977). Barbera and Peleg(1990) showed that strategy−proofness of resolute social choice functions implies Modified strong positive association property (MSPAP). But MSPAP is not equivalent to strategy−proofness. We shall show that strategy−proofness and NNAP are equivalent for resolute social choice functions with continuous preferences.

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1 Introduction

We consider the relation between strategy-proofness of resolute (single-valued) social choice functions (or voting rules) and its property which we call *non-negative association property (NNAP)* when individual preferences over infinite number of alternatives are continuous, and the set of alternatives is a metric space. NNAP is a weaker version of *strong positive association property (SPAP)* of Muller and Satterthwaite (1977)[1)]. Barbera and Peleg (1990) showed that strategy-proofness of resolute social choice functions implies *modified strong positive association property (MSPAP)*. But MSPAP is not equivalent to strategy-proofness. NNAP for social choice functions is the following condition:

> Assume that for two distinct alternatives, x and y, there is an individual preference profile u such that individuals in a group S prefer x to y, individuals in a group S' are indifferent between x and y, individuals in a group S'' prefer y to x and a social choice function chooses x. Consider another profile u' such that individuals in S prefer x to y, individuals in S' prefer x to y or their preferences are identical to those at u, then the social choice function does not choose y at u'.

We shall show that strategy-proofness and NNAP are equivalent for resolute social choice functions with continuous preferences. This result is an extension of the works of Muller and Satterthwaite (1977), Barbera and Peleg (1990) and Tanaka (2001)[2)].

2 Notations and definitions

Notations and terminologies are borrowed from Barbera and Peleg (1990). The set of alternatives is denoted by $\mathscr{A}$ which is a metric space. The metric of $\mathscr{A}$ is denoted by d. $N = \{1, 2, \dots, n\}$ is the finite set of individuals with $n \geqq 2$. The individuals are indexed by individual i, j and so on, and the alternatives are represented by x, y, z and so on. The preference of individual i over the alternatives is represented by $u_i \in U$, where U is the set of continuous real-valued utility

[1)]Muller and Satterthwaite (1977) showed that strategy-proofness and strong positive association property are equivalent for resolute social choice functions with unrestricted domain and linear (strict) individual preferences.

[2)]In Tanaka (2001) we have shown that strategy-proofness is equivalent to *generalized monotonicity*, which is similar to NNAP in this paper, for resolute social choice functions with unrestricted domain and general individual preferences which allow indifference.

functions on $\mathscr{A}$. A profile of the individual preferences is denoted by a *utility profile* such as $u = \{u_1 \cdots, u_n\} \in U^N$, where U^N is the set of all utility profiles for N, and if $u \in U^N$ and $j \in N$ then u_j is the j-th component of u. $\hat{u} = u/u_i^1$ denotes the profile where $\hat{u}_j = u_j$ for all $j \neq i$, and $\hat{u}_i = u_i^1$.

A social choice function (or voting rule) is a function $f : U^N \Rightarrow \mathscr{A}$. When a social choice function chooses x at a profile u, we denote $f(u) = x$. The range of f is denoted by r_f. We call the alternative which is chosen by a social choice function the *winner* of the social choice function. We consider resolute social choice functions which choose only one of the alternatives at every profile, and we assume that r_f has at least three elements.

Suppose that at a utility profile u a social choice function chooses x, and at another profile $\hat{u} = u/u_i^1$ it chooses y. If $u_i^1(x) > u_i^1(y)$ for individual i, the social choice function is manipulable by him at $\hat{u}$ by u_i because he can make the social choice function choose x by misrepresenting a utility function u_i when his true utility function is u_i^1. Similarly, if $u_i(y) > u_i(x)$ for individual i, the social choice function is manipulable by him at u by u_i^1.

Strategy-proofness If a social choice function is not manipulable by any individual at every utility profile, it is *strategy-proof*.

Strong positive association property (SPAP) of Muller and Satterthwaite (1977) and *modified strong positive association property (MSPAP)* of Barbera and Peleg (1990) are defined as follows:.

Strong positive association property (SPAP) A social choice function satisfies SPAP if for every $u \in U^N$, $i \in N$, $u_i^1 \in U$, if $f(u) = x$ and $[u_i(x) \geqq u_i(y) \Rightarrow u_i^1(x) \geqq u_i^1(y)$, for all $y \in r_f]$, then $f(u/u_i^1) = x$.

Modified strong positive association property (MSPAP) A social choice function satisfies MSPAP if for every $u \in U^N$, $i \in N$, $u_i^1 \in U$, if $f(u) = x$ and $[u_i(x) \geqq u_i(y)$ and $y \neq x \Rightarrow u_i^1(x) > u_i^1(y)$, for all $y \in r_f]$, then $f(u/u_i^1) = x$.

SPAP is not necessarily satisfied by a strategy-proof social choice function. Barbera and Peleg (1990) showed that strategy-proof social choice functions satisfy MSPAP. But the converse does not hold as the following example shows.

An example Consider a society with two individuals 1 and 2, and there are four alternatives x, y, z and w. We assume that at a utility profile u, $u_1(x) = u_1(y) = u_1(z) = u_1(w)$, $u_2(z) < u_2(x) < u_2(y) < u_2(w)$ and $f(u) = x$, at another profile

u', $u'_1(x) = u'_1(y) = u'_1(z) = u'_1(w)$, $u'_2(w) < u'_2(y) < u'_2(x) < u'_2(z)$ and $f(u') = y$, and at all other profiles the social choice function chooses one of individual 1's most preferred alternatives. This social choice function does not violate MSPAP because between u and u' $u_2(z) < u_2(x)$ is changed to $u'_2(x) < u'_2(z)$, and between u' and u $u'_2(w) < u'_2(y)$ is changed to $u_2(y) < u_2(w)$. But it is manipulable by individual 2 at u by u'_2, and also manipulable by him at u' by u_2.

Now we define *non-negative association property (NNAP)*.

Non-negative association property (NNAP) Suppose that there is a utility profile $u \in U^N$ such that for alternatives x and y ($x \neq y$)

(1) individuals in a group S ($S \subset N$): $u_i(x) > u_i(y)$

(2) individuals in a group S' ($S' \subset N$, $S' \cap S = \emptyset$): $u_i(x) = u_i(y)$

(3) others (group S''): $u_i(y) > u_i(x)$

and a social choice function chooses x ($f(u) = x$). Let $u' \in U^N$ be a profile such that

(1) individuals in S: $u'_i(x) > u'_i(y)$

(2) individuals in S': $u'_i(x) > u'_i(y)$ or their utility functions do not change ($u'_i(x) = u_i(x)$ for all $x \in \mathscr{A}$)

Then, the social choice function does not choose y at u' ($f(u') \neq y$).

3 Equivalence of NNAP and strategy-proofness

We show the following theorem.

Theorem 1. *Non-negative association property (NNAP) and strategy-proofness for social choice functions are equivalent.*

We prove this theorem by two steps.

Step 1. *Strategy proof social choice functions satisfy NNAP.*

In the following proof we use notations in the above definition of NNAP. This proof is somewhat complicated. Thus we use some graphs.

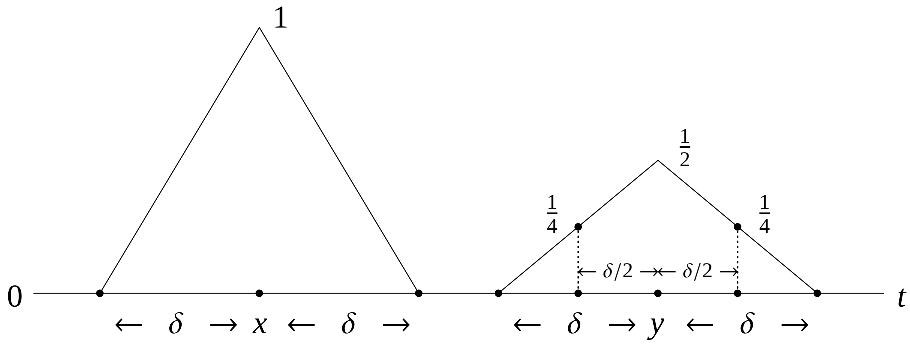

Figure 1: Utility function $v_j(t)$ for individuals in $S \cup S'$

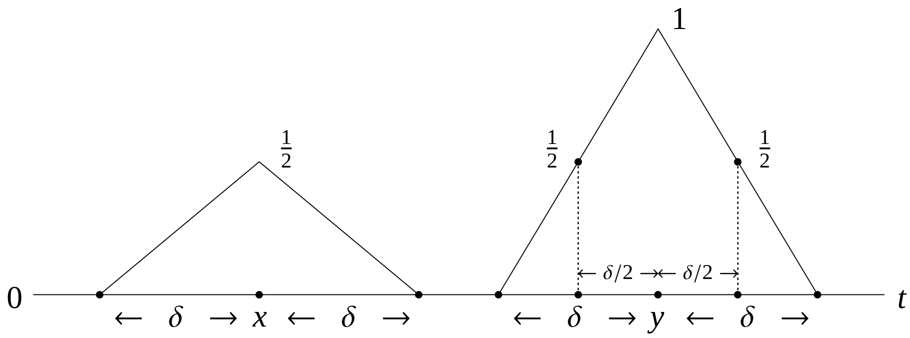

Figure 2: Utility function $v_k(t)$ for individuals in S''

Proof. Let individuals 1 to m ($0 \leq m \leq n$) belong to S, individuals $m + 1$ to m' ($m \leq m' \leq n$) belong to S', and individuals $m' + 1$ to n belong to S''.

(1) Let δ be a positive real number, we define the neighborhoods of x and y by $B(x, \delta) = \{z \in \mathscr{A} | d(x, z) < \delta\}$ and $B(y, \delta) = \{z \in \mathscr{A} | d(y, z) < \delta\}$. Since $\mathscr{A}$ is a metric space and the utility functions are continuous, we can make the value of δ be arbitrarily small such that $B(x, \delta)$ and $B(y, \delta)$ are disjoint, and we have $u_i'(x_0) > u_i'(y_0)$ for $i \in S \cup S'$ and $u_i(y_0) > u_i(x_0)$ for $i \in S''$ for $x_0 \in B(x, \delta)$, $y_0 \in B(y, \delta)$.

For individuals in S and S' we define the following utility function[3)]:

$$v_j(t) = \frac{d(t, \mathscr{A} - B(x, \delta))}{d(t, x) + d(t, \mathscr{A} - B(x, \delta))} + \frac{d(t, \mathscr{A} - B(y, \delta))}{2[d(t, y) + d(t, \mathscr{A} - B(y, \delta))]}$$

[3)] About those functions we are inspired by Barbera and Peleg (1990).

And for individuals in S'' we define the following utility function:

$$v_k(t) = \frac{d(t, \mathscr{A} - B(x,\delta))}{2[d(t,x) + d(t, \mathscr{A} - B(x,\delta))]} + \frac{d(t, \mathscr{A} - B(y,\delta))}{d(t,y) + d(t, \mathscr{A} - B(y,\delta))}$$

They are non-negative and continuous. These utility functions are illustrated in Figure 1 and 2 in the one-dimensional case. Of course we do not assume one dimensional space.

Let us consider a utility profile v such that the utility functions of all individuals in S and S' (denoted by j) are v_j, and the utility functions of all individuals in S'' (denoted by k) are v_k.

Assuming that individual 1 belongs to S, let v^1 be a utility profile such that only the utility function of individual 1 changes from u_1 to v_1, and suppose that at v^1 the social choice function chooses an alternative other than x ($f(v^1) \neq x$). Then, individual 1 has an incentive to report a false utility function u_1 when his true utility function is v_1 because x is the maximal element for v_1, and hence we have $f(v^1) = x$. By the same logic, when the utility functions of individuals 1 to m' change from u_j to v_j (denote such a profile by $v^{m'}$), we have $f(v^{m'}) = x$.

(2) Next, let $v^{m'+1}$ be a utility profile such that the utility function of individual $m'+1$, as well as the utility functions of the first m' individuals, changes from $u_{m'+1}$ to $v_{m'+1}$, and suppose $f(v^{m'+1}) \in B(y,\delta)$. Then, individual $m'+1$ has an incentive to report a false utility function $v_{m'+1}$ when his true utility function is $u_{m'+1}$ because $u_{m'+1}(y_0) > u_{m'+1}(x_0)$ for $x_0 \in B(x,\delta),\ y_0 \in B(y,\delta)$. On the other hand, if $f(v^{m'+1}) \notin B(x,\delta) \cup B(y,\delta)$, individual $m'+1$ has an incentive to report a false utility function $u_{m'+1}$ when his true utility function is $v_{m'+1}$ because $v_{m'+1}(x_0) > v_{m'+1}(z) = 0$ for $x_0 \in B(x,\delta)$ and $z \in \{\mathscr{A} - B(x,\delta) \cup B(y,\delta)\}$. Therefore, we have $f(v^{m'+1}) \in B(x,\delta)$. By the same logic, when the preferences of all individuals change from u_i to v_i, we have $f(v) \in B(x,\delta)$.

(3) Now, suppose that the individual utility functions change one by one from v_i to u'_i. Then, when the utility function of some individual changes, the winner of the social choice function can not change directly from $x_0 \in B(x,\delta)$ to $y^* \in B(y, \frac{\delta}{2})$. If the social choice function chooses $y^* \in B(y, \frac{\delta}{2})$ when the utility function of an individual in $S \cup S'$ (denoted by j) changes from v_j to u'_j, individual j has an incentive to report a false utility function v_j when his true utility function is u'_j because $u'_j(x_0) > u'_j(y^*)$ for $x_0 \in B(x,\delta),\ y^* \in B(y, \frac{\delta}{2})$.

On the other hand, if the social choice function chooses $y^* \in B(y, \frac{\delta}{2})$ when the utility function of an individual in S'' (denoted by k) changes from v_k to u'_k, individual k has an incentive to report a false utility function u'_k when his true utility function is v_k because $v_k(y^*) > v_k(x_0)$ for $x_0 \in B(x, \delta)$, $y^* \in B(y, \frac{\delta}{2})$. Notice $v_k(y^*) > \frac{1}{2}$ and $v_k(x_0) \leqq \frac{1}{2}$. See Figure 2.

(4) It remains the possibility, however, that the winner of the social choice function changes from $x_0 \in B(x, \delta)$ through $w \in \{\mathscr{A} - B(x, \delta) \cup B(y, \frac{\delta}{2})\}$ to $y^* \in B(y, \frac{\delta}{2})$. Suppose that when the utility functions of some individuals have changed from v_i to u'_i, the winner of the social choice function is $w \in \{\mathscr{A} - B(x, \delta) \cup B(y, \frac{\delta}{2})\}$, and further when the utility function of individual l ($l \in S \cup S'$ or $l \in S''$) changes from v_l to u'_l, the winner of the social choice function becomes $y^* \in B(y, \frac{\delta}{2})$. Since $v_l(y^*) > v_l(w)$ for $y^* \in B(y, \frac{\delta}{2})$ and $w \in \{\mathscr{A} - B(x, \delta) \cup B(y, \frac{\delta}{2})\}$, he can get y^* by misrepresenting his utility function u'_l when his true utility function is v_l. Notice $v_j(y^*) > \frac{1}{4}$, $v_j(w) \leqq \frac{1}{4}$, $v_k(y^*) > \frac{1}{2}$ and $v_k(w) \leqq \frac{1}{2}$. See Figure 1 and 2. Therefore, if the social choice function is strategy-proof, in the sequence of changes of individual utility functions the winner of the social choice function does not change from $x_0 \in B(x, \delta)$ through $w \in \{\mathscr{A} - B(x, \delta) \cup B(y, \frac{\delta}{2})\}$ to $y^* \in B(y, \frac{\delta}{2})$, and hence we must have $f(u') \neq y$.

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Next we show the converse of Step 1.

Step 2. *NNAP for social choice functions implies strategy-proofness.*

Proof. Let u be a profile such that a social choice function chooses x ($f(u) = x$), and assume that a social choice function which satisfies NNAP is manipulable. Then, there is a case where, when the utility function of one individual (denoted by i) changes from u_i to, for example, u_i^1 (denote such a profile by u/u_i^1), the winner of the social choice function changes from x to y and we have $u_i(y) > u_i(x)$.

Comparing u and u/u_i^1, individual i prefers y to x at u and the utility functions of other individuals are the same. Thus, those who prefer x to y at u also prefer x to y at u/u_i^1, and the utility functions of individuals who are indifferent between x and y at u do not change from u to u/u_i^1. From NNAP, if the social choice function chooses x at u ($f(u) = x$), it does not choose y at u/u_i^1 ($f(u/u_i^1) \neq y$). Therefore, the social choice function must not be manipulable.

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We have completed the proof of Theorem 1.

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