

Elasticity and revenue: a reappraisal

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Abstract

It seems from what is explained in some textbooks that, for a given demand function, price and total revenue move in the same direction when elasticity is smaller than one and move in opposite directions when elasticity is greater than one, with elasticity from point (p, q) to point (p', q') of a demand function defined as $-[(q' - q) / q] / [(p' - p) / p]$. Since these two results turn out to be false, this comment clarifies the relationship between elasticity (as previously defined), price movements and changes in total revenue.

This paper is the result of the following allocation of tasks: Ted Bergstrom did the part that is interesting, correct and worth reading, whereas I did the rest.

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1. Introduction

When one is interested in defining a price elasticity measure between two arbitrary points of a demand function, the concept of price elasticity of demand at a point is not useful¹. Textbooks then offer two possible measures: the price elasticity from one point to another and the arc price elasticity between two points.

Let $a = (p_0, q_0)$ and $b = (p_1, q_1)$ be two arbitrary points of a demand function. Define $\Delta p = p_1 - p_0$, $\Delta q = q_1 - q_0$, $p^* = (p_0 + p_1) / 2$ and $q^* = (q_0 + q_1) / 2$. The price elasticity of demand ϵ from a to b is $\epsilon = - (\Delta q / q_0) / (\Delta p / p_0)$, whereas the arc price elasticity of demand α between a to b is $\alpha = - (\Delta q / q^*) / (\Delta p / p^*)$. As in the elasticity at a point case, it would be desirable to have an unambiguous relationship between elasticity values, price changes and changes in total revenue. In this respect, the treatment of the elasticity concept ϵ in many introductory and intermediate textbooks² may lead the student to believe that rules R1 and R2 are true.

R1. If $\epsilon < 1$ then the price change and the change in total revenue have moved in the same direction.

R2. If $\epsilon > 1$ then the price change and the change in total revenue have moved in opposite directions.

Examples 1 and 2 prove R1 and R2 to be false.

Example 1. Let $(p_0, q_0) = (1, 10)$ and $(p_1, q_1) = (2, 4)$. In this case, $\epsilon = 3/5 < 1$ and the price has increased but total revenue has not increased, which disproves R1.

¹ If the term “infinitesimal change” has any economic meaning at all, the elasticity at a point concept is relevant for “infinitesimal changes”. Unfortunately, it is not always made clear the scope of usefulness of the concept. For instance, in results (a), (b) and (c), McKenna and Rees (1992, p. 25) speak of “changes in price” and “reduction in price”, as if they could be arbitrary.

² It could be inferred that R1 and R2 hold true (at least, for “small” price changes) from what is explained in, for instance: (i) Miller (1982, ch. 6); (ii) Miller and Meiners (1986, pp. 158-159); (iii) Chacholiades (1986, pp. 25-26); (iv) Wonnacott and Wonnacott (1986, ch. 20); (v) Fischer, Dornbusch and Schmalensee (1988, sec. 5.2); (vi) Lipsey (1989, ch. 6); (vii) Eaton and Eaton (1991, p. 94); (viii) Frank (1991, p. 145); (ix) Hirshleifer and Glazer (1992, p. 123); (x) Stiglitz (1994, ch. 5); (xi) Hardwick, Khan and Langmead (1994, ch. 3); (xii) Bowden and Bowden (1995, pp. 393-394); (xiii) Lipsey and Chrystal (1995, ch. 5); (xiv) Varian (1996, sec. 15.7); (xv) Mankiw (1998, ch. 5); (xvi) Mochón (2000, pp. 75-76); and (xvii) Pindyck and Rubinfeld (2001, sec. 4.3).

Example 2. Let $(p_0, q_0) = (10, 1)$ and $(p_1, q_1) = (4, 2)$. In this case, $\varepsilon = 5/3 > 1$ and the price has diminished but total revenue has not increased, which disproves R2.

It may appear that R1 and R2 fail in these examples because the price changes are not “small”. This intuition is not correct. Section 2 clarifies from an algebraic point of view the relationship between price elasticity of demand ε , price changes and variations in total revenue, while Section 3 clarifies this relationship from a geometric point of view. Finally, Section 4 shows that there are elasticity concepts defined for arbitrary price changes for which R1 and R2 hold. The elasticity concept α is an example.

2. Price elasticity of demand and total revenue: the algebra

Let $a = (p_0, q_0)$ and $b = (p_1, q_1)$ be two different points of a strictly decreasing demand function defined for positive prices and whose values are positive. Reproducing a development by Ted Bergstrom, define $\Delta p = p_1 - p_0$, $\Delta q = q_1 - q_0$ and $\Delta R = p_1 q_1 - p_0 q_0$ so that the price elasticity of demand from a to b is $\varepsilon = -(\Delta q / q_0) / (\Delta p / p_0) = -p_0 \Delta q / q_0 \Delta p$. Since $\Delta R = p_1 q_1 - p_0 q_0 = q_1 \Delta p + p_0 \Delta q$, it follows that $\Delta R / \Delta p = q_1 + (p_0 \Delta q / \Delta p) = q_1 - \varepsilon q_0$. Therefore, recalling that either $\Delta p > 0$ or $\Delta p < 0$,

$$\Delta R / \Delta p = q_0(1 - \varepsilon + \Delta q / q_0). \quad (1)$$

The relationship between price elasticity of demand and changes in total revenue caused by price changes can be easily established from (1). For instance, (2)-(5) make apparent what can and what cannot be inferred from the fact that $\varepsilon < 1$, whereas (6)-(9) make apparent what can and what cannot be inferred from the fact that $\varepsilon > 1$. Observe that Example 1 proves both (3) and (5), whereas Example 2 proves both (7) and (9).

$$\varepsilon < 1 \Rightarrow [\Delta R > 0 \Rightarrow \Delta p > 0] \quad (2)$$

$$\varepsilon < 1 \not\Rightarrow [\Delta p > 0 \Rightarrow \Delta R > 0] \quad (3)$$

$$\varepsilon < 1 \Rightarrow [\Delta p < 0 \Rightarrow \Delta R < 0] \quad (4)$$

$$\varepsilon < 1 \not\Rightarrow [\Delta R < 0 \Rightarrow \Delta p < 0] \quad (5)$$

$$\varepsilon > 1 \Rightarrow [\Delta R > 0 \Rightarrow \Delta p < 0] \quad (6)$$

$$\varepsilon > 1 \not\Rightarrow [\Delta p < 0 \Rightarrow \Delta R > 0] \quad (7)$$

$$\varepsilon > 1 \Rightarrow [\Delta p > 0 \Rightarrow \Delta R < 0] \quad (8)$$

$$\varepsilon > 1 \not\Rightarrow [\Delta R < 0 \Rightarrow \Delta p > 0] \quad (9)$$

To illustrate how (2)-(9) follow from (1), consider (2) and (3). As regards (2), suppose that $\varepsilon < 1$ and $\Delta R > 0$ but $\Delta p < 0$. Thus, the left side of (1) is negative. Since $\Delta p < 0$ implies $\Delta q > 0$ and since $1 - \varepsilon > 0$, the right side of (1) is positive: contradiction. With respect to (3), $\varepsilon < 1$ and $\Delta p > 0$ (which implies $\Delta q < 0$) do not ensure that $(1 - \varepsilon + \Delta q / q_0)$ is positive: a percentage change $\Delta q / q_0$ sufficiently high (in absolute terms) may make the right side of (1) negative, so that $\Delta R / \Delta p < 0$ and $\Delta R < 0$.

Equation (1) also helps to clarify what can and what cannot be inferred from a change in price. In particular, it is easy to prove (10)-(17) from (1).

$$\Delta p > 0 \Rightarrow [\Delta R > 0 \Rightarrow \varepsilon < 1] \quad (10)$$

$$\Delta p > 0 \not\Rightarrow [\varepsilon < 1 \Rightarrow \Delta R > 0] \quad (11)$$

$$\Delta p > 0 \Rightarrow [\varepsilon > 1 \Rightarrow \Delta R < 0] \quad (12)$$

$$\Delta p > 0 \not\Rightarrow [\Delta R < 0 \Rightarrow \varepsilon > 1] \quad (13)$$

$$\Delta p < 0 \Rightarrow [\Delta R > 0 \Rightarrow \varepsilon > 1] \quad (14)$$

$$\Delta p < 0 \not\Rightarrow [\varepsilon > 1 \Rightarrow \Delta R > 0] \quad (15)$$

$$\Delta p < 0 \Rightarrow [\varepsilon < 1 \Rightarrow \Delta R < 0] \quad (16)$$

$$\Delta p < 0 \not\Rightarrow [\Delta R < 0 \Rightarrow \varepsilon < 1] \quad (17)$$

It is finally worth noticing that it is not possible to ascertain whether $\varepsilon > 1$ or $\varepsilon < 1$ from the fact that a change in the price in a certain direction causes a change in total revenue in a certain direction. In particular, having price and total revenue move in the same direction does not guarantee that $\varepsilon > 1$ and, in addition, having price and total revenue move in opposite directions does not guarantee that $\varepsilon < 1$. More specifically, it follows from (1) that: (i) $[\Delta p > 0 \Rightarrow \Delta R > 0] \not\Rightarrow \varepsilon < 1$; (ii) $[\Delta p < 0 \Rightarrow \Delta R < 0] \not\Rightarrow \varepsilon < 1$; (iii) $[\Delta p < 0 \Rightarrow \Delta R > 0] \not\Rightarrow \varepsilon > 1$; and (iv) $[\Delta p > 0 \Rightarrow \Delta R < 0] \not\Rightarrow \varepsilon > 1$.

3. Price elasticity of demand and total revenue: the geometry

The aim of this section is to present a geometric analysis of the relationship between ε , Δp and ΔR . Set $Q = q_1/q_0$ and $P = p_1/p_0$. Notice that $P > 1$ implies $Q < 1$ and that $P < 1$ implies $Q > 1$. Consider first the case in which $\Delta p > 0$. This is equivalent to having $P > 1$ and $Q < 1$, so regions A, B and C in Figure 1 represent the fact that $\Delta p > 0$; that is, if $\Delta p > 0$ occurs then the values of P and Q can only be found in regions A, B or C. When $\Delta p > 0$, it can be easily verified that $(\varepsilon < 1 \Leftrightarrow P + Q > 2)$ and $(\varepsilon > 1 \Leftrightarrow P + Q < 2)$. As a result, when $\Delta p > 0$, B and C represent $\varepsilon < 1$, whereas A represents $\varepsilon > 1$. Finally,

given that $(\Delta R > 0 \Leftrightarrow PQ > 1)$ and $(\Delta R < 0 \Leftrightarrow PQ < 1)$, if $\Delta p > 0$ then A and B represent $\Delta R < 0$ and C represents $\Delta R > 0$.

The case in which $\Delta p < 0$ is similar. Now, $\Delta p < 0$ is equivalent to $P < 1$ and $Q > 1$, so D , E and F in Figure 1 represent $\Delta p < 0$. If $\Delta p < 0$ then $(\epsilon < 1 \Leftrightarrow P + Q < 2)$ and $(\epsilon > 1 \Leftrightarrow P + Q > 2)$. Accordingly, when $\Delta p < 0$, E and F represent $\epsilon > 1$ and D represents $\epsilon < 1$. Lastly, if $\Delta p < 0$ then D and E represent $\Delta R < 0$ and F represents $\Delta R > 0$.

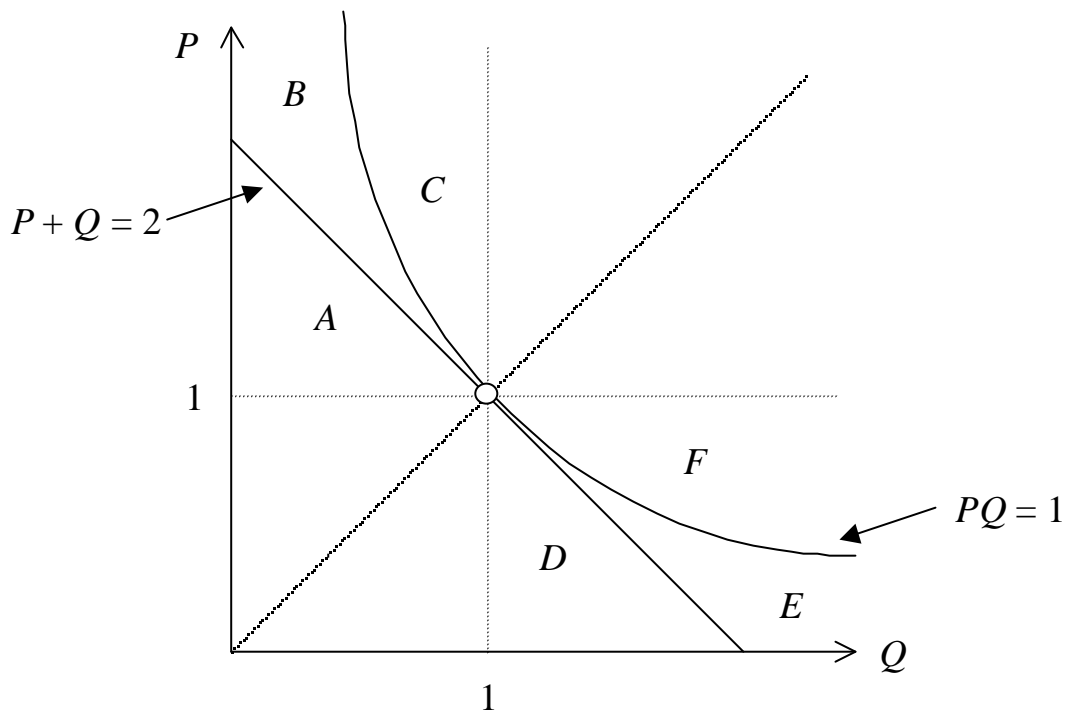


Fig. 1

With this information, (2)-(17) can be illustrated and proved geometrically by means of Figure 1. Consider, for instance, (2) and (3). As regards (2), suppose $\epsilon < 1$ and $\Delta R > 0$. The regions representing $\epsilon < 1$ are B , C and D , while those representing $\Delta R > 0$ are C and F . Consequently, the only common region is C , where $\Delta p > 0$ occurs. With respect to (3), suppose $\epsilon < 1$ and $\Delta p > 0$. The regions consistent with $\epsilon < 1$ are B , C and D . Those consistent with $\Delta p > 0$ are A , B and C . The intersection is given by B and C , where $\Delta R > 0$ does not necessarily occur: all points in B (which satisfy $P + Q > 2$ and $PQ < 1$) disprove the statement “ $\epsilon < 1$ and $\Delta p > 0$ imply $\Delta R > 0$ ”. Observe that the corresponding P and Q in Example 1 lie in B . Observe as well that no matter how close P is to 1 from above (that is, how close $p_1 > p_0$ is to p_0), one can still find some Q in region B .

4. Concluding comments

Some introductory and intermediate textbooks do not introduce both ε and α , but only introduce the latter³. This is the approach followed, for instance, by Samuelson and Nordhaus (1992), Schiller (1993) and Pashigian (1995). In a footnote, Pashigian (1995, sec. 1.6) even proves that R2 holds for α . To see that both R1 and R2 hold for α , note first that $p_1q_0 - p_0q_1 = q_0\Delta p - p_0\Delta q = q_0\Delta p(1 - p_0\Delta q/q_0\Delta p) = q_0\Delta p(1 + \varepsilon)$. Hence, $\alpha = -(\Delta q/\Delta p)(p^*/q^*) = -(p_0\Delta q + p_1\Delta q)/(q_0\Delta p + q_1\Delta p) = (p_1q_0 - p_0q_1 - p_1q_1 + p_0q_0)/(p_1q_0 - p_0q_1 + p_1q_1 - p_0q_0) = (p_1q_0 - p_0q_1 - \Delta R)/(p_1q_0 - p_0q_1 + \Delta R) = (q_0\Delta p(1 + \varepsilon) - \Delta R)/(q_0\Delta p(1 + \varepsilon) + \Delta R)$. As a consequence,

$$\alpha = \frac{q_0(1 + \varepsilon) - \Delta R/\Delta p}{q_0(1 + \varepsilon) + \Delta R/\Delta p}. \quad (18)$$

It follows from (18) that ($\alpha < 1 \Leftrightarrow \Delta R/\Delta p > 0$) and ($\alpha > 1 \Leftrightarrow \Delta R/\Delta p < 0$). Thus, when $\alpha < 1$, price changes and changes in total revenue move in the same direction and, when $\alpha > 1$, price changes and changes in total revenue move in opposite directions.

In view of this result, α could be deemed a more satisfactory elasticity definition than ε . The problem then lies in the arbitrariness of choosing the average as the value against to which measure absolute changes. This consideration leads to the following open problem: identify those merging real-valued functions $f(x, y)$ such that the elasticity concept $e = -(\Delta q/f(q_0, q_1))/(\Delta p/f(p_0, p_1))$ satisfies both R1 and R2. Two examples of such functions are $f(x, y) = (x + y)/2$ and $f(x, y) = x + y$.

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³ None of those previously mentioned textbooks in which both elasticity concepts are introduced clarify whether R1 and R2 hold for one or for both concepts. The shift in Lipsey (1989), Lipsey and Chrystal (1995) or Mankiw (1998) from an initial presentation of the elasticity from one point to another to the arc elasticity between two points (by claiming that using average values is a better “method” to compute elasticity values) adds more confusion: if this is the case, it seems better to introduce only arc elasticities.

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