E C O N O M I C S B U L L E T I N

An Atkinson−Gini family of social evaluation functions

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Abstract

We investigate the properties of a family of social evaluation functions and inequality indices which merge the features of the family of Atkinson (1970) and S−Gini (Donaldson and Weymark (1980, 1983), Yitzhaki (1983) and Kakwani (1980)) indices. Income inequality aversion is captured by decreasing marginal utilities, and aversion to rank inequality is captured by rank−dependent ethical weights, thus providing an ethically−flexible dual basis for the assessment of inequality and equity. These social evaluation functions can be interpreted as average utility corrected for the illfare of relative deprivation. They can alternatively be understood as averages of altruistic well−being in a population. They moreover have a simple graphical interpretation.

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1 Introduction

We investigate the properties of a family of social evaluation functions and of inequality indices which merge the features of two popular classes of inequality indices. Our social evaluation functions, denoted by $W_{\rho,\epsilon}$, are indeed a combination of the family of Atkinson (1970) indices, characterised by a normative parameter ϵ of aversion to income inequality, and of the family of S-Gini (or Single-parameter Gini) indices of Donaldson and Weymark (1980, 1983) and Yitzhaki (1983) (see also Kakwani (1980)), characterised by an analogous normative parameter ρ of aversion to rank inequality.

The Atkinson family of social evaluation functions is traditionnally linked to utilitarianism and to expected utility theory in the risk literature; the cost of risk and inequality is captured by decreasing marginal utilities of income. The S-Gini social evaluation functions come from a generalisation of the most common index of inequality, the Gini index, and emphasize the importance of ranks and interpersonal comparisons in making social welfare assessments. The ethical criteria of our own social evaluation functions correspondingly rely on the use of decreasing (individual or social) marginal utilities of incomes to capture the dispersion of incomes around their mean value, and on the use of rank-dependent ethical weights to capture the dispersion of ranks in a population.

The link of these social evaluation functions with both classical utilitarianism and rank-based measures allows one easily to check the ethical sensitivity to rank and income dispersion in measuring overall inequality. It also makes the social evaluation functions amenable to the study of features of equity which are variably dependent on either of these two aspects of dispersion. This is the case, for instance, of the study of horizontal inequity, which can be concerned either with the tax-induced dispersion of incomes at a given rank in the population, or with the changes in ranks induced by a tax and benefit system.

We introduce the general formulation of our social evaluation functions in Section 2. This general formulation appears utilitarian in format, but differs from the traditional utilitarian approach through the use of rank-dependent weights on the utilities. For these social evaluation functions to fulfill the axioms which characterise the S-Gini indices and for them to yield relative inequality indices, they must further take the particular form of $W_{\rho,\epsilon}$. The associated relative inequality indices are also defined, in a discrete and in a continuous setting. Section 3 then shows how the social evaluation functions can be interpreted as average utility corrected for relative deprivation in individual utility. Section 4 also links the social evaluation functions to averages of altruistic well-being in the population. In other words, the social evaluation functions take into account interpersonal comparisons in a way which can be interpreted either as resentments for one's relative deprivation, or as concerns for the welfare of others. Section 5 provides a brief graphical interpretation of the indices, and Section 6 concludes. The appendix contains the proofs of all of the propositions.

2 A class of social evaluation functions

For the discrete setting, we suppose that there are n individuals in the population, with (positive) incomes denoted by y_i, and ordered such that $y_1 \geq y_2 \geq ... \geq y_{n-1} \geq y_n$. Let $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ be the vector of incomes; hence $\mathbf{y} \in \mathcal{R}^n_+$, the $n-$dimensional positive orthant. A general form for a "utilitarian-looking" social evaluation function can then be defined as:

$$W^n(\mathbf{y}) = \frac{\sum_{i=1}^n g_i^n U(y_i)}{\sum_{i=1}^n g_i^n} \tag{1}$$

where $U(y)$ is interpreted as a utility function that is continuous and increasing in y and where g_i^n is a positive weight applied on the utility $U(y_i)$ of individual i. Without loss of generality, we normalise the first weight, g_1^n, to 1 throughout the paper.

As is conventional in the literature, we can define an equally distributed equivalent income (or EDE income, which is a money-metric measure of social welfare) $\xi^n(\mathbf{y}) : \mathcal{R}_+^n \to \mathcal{R}$, as:

$$W^n(\xi^n(\mathbf{y}) \cdot \mathbf{1}) = W^n(\mathbf{y}) \tag{2}$$

where $\mathbf{1}$ is a $n-$dimensional vector of 1's. From (1), the explicit expression for $\xi^n(\mathbf{y})$ is

$$\xi^n(\mathbf{y}) = U^{-1}(W^n(\mathbf{y})) \tag{3}$$

with $U^{-1}(\cdot)$ being the inverse utility function. From this, we can follow convention and define for (1) an index of inequality $I^n(\mathbf{y})$ as:

$$I^n(\mathbf{y}) = 1 - \frac{\xi^n(\mathbf{y})}{\mu(\mathbf{y})} \tag{4}$$

where $\mu(\mathbf{y}) = n^{-1} \sum_{i=1}^n y_i$ is the arithmetic mean.

$$g_1^n \leq g_2^n \leq \cdots \leq g_n^n. \tag{5}$$

We focus in this paper on social evaluation functions which yield relative inequality indices. An index of inequality is said to be relative if and only if

$$I^n(\lambda \mathbf{y}) = I^n(\mathbf{y}) \tag{6}$$

for all $\lambda > 0$ and for all $\mathbf{y} \in \mathcal{R}_+^n$.

Formulation (1) has a clear utilitarian flavour, but differs from the traditional utilitarian specification by the presence of the rank-dependent weights g_i^n. The formulation also replaces income by income utility in the locally income linear — or generalised Gini — formulations of Donaldson and Weymark (1980) and Mehran (1976). Generalised Ginis are defined by $I^n(\mathbf{y})$ in equation (4) with $U(y) = y$. The formulation is also linked to rank-dependent expected utility theory, as noted in Chew and Epstein (1989) and Ben Porath and Gilboa (1994).

As Sen (1973, p.39) argues, $U(y_i)$ can be an individual utility function, or it can be the "component of social welfare corresponding to person i, being itself a strictly concave function of individual utilities". Sen also adds that "it is fairly restrictive to think of social welfare as a sum of individual welfare components" (p.39), and that one might feel that "the social value of the welfare of individuals should depend crucially on the levels of welfare (or incomes) of others" (p.41). As we will see clearly later, the formulation of equation (1) allows precisely for this by applying rank-dependent weights on each individual utility component $U(y_i)$. Only in the simple case of g_i^n being a constant across i do we obtain the traditional utilitarian formulation. Moreover, as Ben Porath and Gilboa (1994, p.445) note, "the most salient drawback of linear measures [*i.e.*, generalised Ginis] is that the effect on social welfare of a transfer of income from one individual to another depends only on the ranking of the incomes but not on their absolute levels". This drawback of linear measures is avoided by the more general formulation of (1). (1) escapes this drawback since $U(y)$ does not have to be affine in incomes. Only when the marginal utility of income $U'(y)$ is constant across y is the social value of a mean-preserving transfer independent of the value of the incomes of the transfer recipient and transfer giver.

S-Gini social evaluation functions are members of single-series Ginis, which are themselves members of the generalised Gini class (see Donaldson and Weymark (1980)). These classes of indices share interesting properties. Mehran (1976) shows that the generalised Gini inequality indices can be easily graphically interpreted as weighted areas between Lorenz curves and lines of perfect equality. Weymark (1981) shows that the (absolute version of the) class of generalised Gini indices is the only one which obeys an axiom of weak independence of income source ("when the distribution of income from all but one source of income is the same in two distributions, overall inequality is determined by the inequality of the last source"). Ben Porath and Gilboa (1994) and Weymark (1995) also show that the class of generalised Gini indices is the only one which obeys an axiom of order-preserving-transfer. This axiom requires for our purposes that a common transfer of individual utility $U(y)$ made simultaneously in two distributions between pairs of individuals who occupy adjacent ranks in the income distributions should preserve the pre- and post-transfer social evaluation ranking of the distributions. Blackorby et al. (1994) demonstrate that the members of the class of generalised Gini indices provide a class of solutions to cooperative bargaining — solutions which respond by the same constant to a constant addition to one agent's component in the feasible set of utility vectors.

For the subclass of single-series Ginis, the weights on the individual $U(y_i)$ arranged in decreasing order are independent of population size, that is, $g_i^n = g_i$. This leads to the social evaluation functions of the following type W:

$$W(\mathbf{y}) = \frac{\sum_{i=1}^{n} g_i U(y_i)}{\sum_{i=1}^{n} g_i}. \tag{7}$$

Bossert (1990) shows that this property is needed if an axiom of separability of the well-being of the rich from the rest of the population is to be obeyed.

S-Ginis form the only subclass of single-series Ginis to satisfy the Dalton Population Principle, by which the addition to a population of an exact replica of that population should not change the social evaluation function. Let $\mathbf{y}^q$ be a $q-$fold replica of $\mathbf{y}$:

$$\mathbf{y}^q = (\underbrace{\mathbf{y}, \cdots, \mathbf{y}}_{q \text{ times}}). \tag{8}$$

W satisfies the Dalton Population Principle if $W(\mathbf{y^q}) = W(\mathbf{y})$, for all $\mathbf{y} \in \mathcal{R}_+^n$ and for all $q = 1, 2, ...$ If we add to this requirement the one of yielding Lorenz-consistent relative inequality indices, we are led to a particular form for the social evaluation functions defined in (7), as shown in Proposition 1.

Proposition 1 *The social evaluation functions W defined in equation (7) (with $g_1 = 1$) are increasing in y_i, obey the principle of population and yield a Lorenz-consistent relative inequality index if and only if*

$$g_i = g_i(\rho) \equiv i^\rho - (i-1)^\rho \tag{9}$$

and $U(y)$ is given by the specific form $U_\epsilon(y)$

$$U_\epsilon(y) \equiv \begin{cases} a + b\frac{y^{1-\epsilon}}{(1-\epsilon)} & \textit{if} \;\; \epsilon \neq 1 \\ a + b\ln(y) & \textit{if} \;\; \epsilon = 1 \end{cases} \tag{10}$$

and $\rho \geq 1$, $\epsilon \geq 0$ and $b > 0$.

Proof: See the appendix.

Note that $\sum_{i=1}^{n} g_i(\rho) = n^\rho$. The result of Proposition 1 thus leads to the following family of social evaluation functions:

$$W_{\rho,\epsilon}(\mathbf{y}) = \sum_{i=1}^{n} \frac{[i^\rho - (i-1)^\rho]U_\epsilon(y_i)}{n^\rho}. \tag{11}$$

Berrebi and Silber (1981, p.393) have in fact already proposed two decades ago a generalised version of this form, although they did not at the time investigate its properties[1]. $W_\rho(\mathbf{y})$ is obtained by replacing $U_\epsilon(y)$ in (11) by the general form $U(y)$. The families of EDE incomes and inequality indices corresponding to $W_{\rho,\epsilon}(\mathbf{y})$ are denoted by $\xi_{\rho,\epsilon}(\mathbf{y})$ and $I_{\rho,\epsilon}(\mathbf{y})$. We can draw on well-known results to state that

$$\frac{\partial \xi_{\rho,\epsilon}(\mathbf{y})}{\partial \rho} \leq 0; \; \frac{\partial I_{\rho,\epsilon}(\mathbf{y})}{\partial \rho} \geq 0; \tag{12}$$

$$\frac{\partial \xi_{\rho,\epsilon}(\mathbf{y})}{\partial \epsilon} \leq 0; \; \frac{\partial I_{\rho,\epsilon}(\mathbf{y})}{\partial \epsilon} \geq 0. \tag{13}$$

This says that the higher the aversion to income inequality, or the higher the aversion to rank inequality, the lower is the EDE income and the higher is the inequality index.

The dual-parameter structure of $W_{\rho,\epsilon}$ thus offers extra ethical flexibility in the specification of the type of inequality aversion that may be of concern to an analyst. Duclos, Jalbert and Araar (2000) also show that this class of social evaluation functions can provide an ethically flexible decomposition of the total redistributive effect of taxes and transfers into a vertical equity effect and a horizontal equity one. Drawing on the dual dimension of the social evaluation functions $W_{\rho,\epsilon}$ also allows a synthesis of two approaches to the measurement of horizontal inequity, the reranking (see Feldstein (1976), Atkinson (1979) and Plotnick (1981)) and the classical approaches (see for instance Duclos and Lambert (2000))[2].

For the continuous setting, we denote by $y_F(p)$ the p-quantile of the distribution of income. $y_F(p)$ is the left inverse of the distribution function $p = F(y)$, defined by $y_F(p) = \inf\{s > 0 | F(s) \geq p\}$ for $p \in [0,1]$, and can be thought of as the income of the p-ranked individual. As for equation (13) in Donaldson and Weymark (1983), $W_\rho(\mathbf{y})$ for a discrete distribution corresponds to the following $W_\rho(F)$ for a continuous distribution:

$$W_\rho(F) = \int_0^1 w(p,\rho)U(y_F(p))dp \tag{14}$$

[1] See Wang and Tsui (2000) and Aaberge (2000) for recent surveys and formulations of other rank-based social evaluation functions.

[2] See also Aronson, Johnson and Lambert (1994) for an influential attempt to consider jointly these two approaches.

where $w(p,\rho) = \rho(1-p)^{\rho-1}$, an ethical weight on individual utility that depends on the individual's rank p and on the parameter ρ. This is also the formula found in Kakwani (1980) and Yitzhaki(1983) when $U(y) = y$. $\xi_\rho(F)$ and $I_\rho(F)$ are defined accordingly. Replacing $U(y)$ by $U_\epsilon(y)$ in (14), we obtain $W_{\rho,\epsilon}(F)$ as a special case of $W_\rho(F)$.

Integration by parts of equation (14) yields an alternative method for computing the social evaluation functions (and thus their EDE incomes and associated indices of inequality):

$$W_\rho(F) = \int_0^1 k(p,\rho) GL_F^U(p)dp \tag{15}$$

where $k(p,\rho) = \rho(\rho-1)(1-p)^{\rho-2}$, and where $GL_F^U(p) = \int_0^p U(y_F(q))dq$ is the generalised Lorenz curve (see Shorrocks (1983)) of utilities. It can also be shown that a convenient way to compute these indices is through a simple covariance formula:

$$W_\rho(F) = \mu_U + \rho \operatorname{cov}(U(y_F(p)), (1-p)^{\rho-1}) \tag{16}$$

where $\mu_U = \int_0^1 U(y_F(p))dp$ is average utility. The second term (the covariance between utilities and a decreasing function of ranks) is negative, and captures the loss of social welfare due to inequality in utility – the term is equal to the distance between average (μ_U) and expected ($W_\rho(F)$) rank-weighted utility.

Note that the estimation of the functions (14) (as well as that of numerous other indices of inequality, social welfare, poverty and redistribution) using sample data can be done using a free and user-friendly software, DAD, that is available at www.mimap.ecn.ulaval.ca. Sampling weights can easily be incorporated in the calculations. This software also calculates the asymptotic sampling distribution of these and other indices.

3 Relative deprivation

We now show how the social evaluation functions introduced above can serve to incorporate interpersonal comparisons of utility in the assessment of social welfare. Such interpersonal comparisons have long been of concern in the socio-psychological literature, which shows that exclusion and interpersonal differences have an impact both on individual well-being and on social cohesion and social welfare[3]. In particular, the theory of relative deprivation suggests that people specifically compare their individual fortune with that of others in establishing their own degree of satisfaction with their own lives.

In the words of Runciman (1966) (an important contributor to that theory), ”the magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the person desiring it” (p.10). Here, we follow Yitzhaki's (1979) and Hey and Lambert's (1980) lead and define for each individual an indicator of relative deprivation which measures the distance between his welfare and that of those towards whom he feels deprived, namely, those whose situation he “desires”. As opposed to Yitzhaki (1979) and Hey and Lambert (1980), however, we use utility and not income to measure individual welfare. Hence, let $\delta(p_i, p_j)$ represent the relative utility

[3]For more specific references to this literature, see for instance Duclos (1998).

deprivation of an individual at rank p_i in the distribution of income, when comparing himself with an individual at rank p_j in the same distribution:

$$\delta_F(p_i, p_j) = \max\left[0,\ U(y_F(p_j)) - U(y_F(p_i))\right]. \tag{17}$$

This says that no relative deprivation is felt by i when he compares himself to an individual j that is less well-off then he is. Otherwise, relative deprivation is captured by $U(y_F(p_j)) - U(y_F(p_i))$. Aggregating this relative deprivation over all individuals j, we find the following expected relative deprivation $d(p_i)$ for the individual at rank p_i:

$$d_F(p_i) = \int_0^1 \delta_F(p_i, p)dp. \tag{18}$$

If we then take an average of $d_F(p_i)$ across all individuals i, and weight each such expected deprivation by the ethical weight $k(p_i, \rho)$, we find $D_\rho(F)$:

$$D_\rho(F) = \frac{1}{\rho}\int_0^1 d_F(p)k(p, \rho)dp. \tag{19}$$

This leads to the following proposition:

Proposition 2 *The social evaluation functions W_ρ can be interpreted as average utility corrected by average relative deprivation in utility:*

$$W_\rho(F) = \mu_U(F) - D_\rho(F). \tag{20}$$

Proof: See the appendix.

The expression D_ρ in (20) being identical by definition to the covariance term in (16), we also find that the cost of inequality in F could thus be interpreted as an ethically weighted average of individual relative deprivation.

4 Altruism

We can alternatively interpret the social evaluation functions W_ρ as averages of altruistic well-being functions. Let an individual at rank p randomly observe $\rho - 1$ other individuals in the population. Denote the incomes of these random individuals by $y(p_1), y(p_2), ..., y(p_{\rho-1})$. Let the enlarged altruistic well-being function of an individual at rank p equal his egoistic utility function $U(y(p))$ plus an altruistic utility component. Think of this altruistic utility as expressing a concern for the well-being of those with low utilities among the $\rho - 1$ individuals that an individual randomly observes. More precisely, define the altruistic well-being component function as the difference between the egoistic utility function $U(y(p))$ and the minimum of the utilities of the other $\rho - 1$ individuals, when the difference is positive. Denoting this difference as $\alpha_F(p, \rho)$, we have:

$$\alpha_F(p, \rho) = U(y_F(p)) - \min\left[U(y_F(p_1)), .., U(y_F(p_{\rho-1}))\right]. \tag{21}$$

We then obtain $a_F(p, \rho)$ as the altruistic well-being component:

$$a_F(p, \rho) = \max\left[0, \alpha_F(p, \rho)\right]. \tag{22}$$

Total utility at rank p is then

$$A_F(p,\rho) = U(y_F(p)) - a_F(p,\rho). \tag{23}$$

This leads to the following proposition:

Proposition 3 *The social welfare function $W_\rho(F)$ is the average of altruistic well-being in the population:*

$$W_\rho(F) = \int_0^1 A_F(p,\rho)dp. \tag{24}$$

Proof: See the appendix.

5 Graphical interpretation

Yitzhaki (1983, p.264) shows how each of the two families incorporated in equation (11), the Atkinson and the S-Gini indices, have a common dual graphical interpretation as a weigthed distance of the cumulative distribution function curve from the ordinate (for the Atkinson indices) and from the abscissa (for the S-Ginis). Figure 1 generalises this interpretation for the general formulation of equation (11). Population ranks p are shown on the horizontal axis, and the utility quantiles $U(y(p)) = y(p)^{1-\epsilon}/(1-\epsilon)$ are shown on the vertical axis. The curve is thus the inverse distribution function of utilities. The contribution of each individual i in the computation of $W_{\rho,\epsilon}$ is the area of the rectangle of height $U(y(p_i))$ – the individual's utility – and of length $\rho(1-p)^{\rho-1}$, a distance between his rank p_i and the top rank (1). The larger the area of the rectangle, the greater the contribution of the individual to social welfare. Social welfare is then simply the average area of all such individual rectangles. When $\rho = 2$, social welfare is twice the average size of the rectangles of the type shown in Figure 1. The traditional Gini social evaluation function, $W_{2,0}$, equals twice the size of all of these rectangles when their height is simply $y(p)$. The traditional Atkinson social evaluation function, $W_{1,\epsilon}$ is the simple integral of the height of the rectangles, $(1-\epsilon)^{-1}\int_0^1 y(p)^{1-\epsilon}dp$. The more averse we are to rank inequality, the more concerned we are about $(1-p)$ in weighting the individual utilities. Loosely speaking, for a given ρ and ϵ, social welfare is largest when there exists no negative correlation between the vertical and horizontal lengths of the individual rectangles. When such a correlation equals zero, $W_{\rho,\epsilon}$ reaches a maximum value of $\mu^{1-\epsilon}/1-\epsilon$.

6 Conclusion

We have proposed a family of social evaluation functions $W_{\rho,\epsilon}$ and of associated inequality indices which merges the features of the family of Atkinson and S-Gini indices. Parameters ϵ of aversion to income inequality and ρ of aversion to rank inequality characterise the individual members of that family. The family of social evaluation functions is shown to be the only one to obey a set of popular axioms in the income distribution literature. The functions can be interpreted as averages of utility corrected for relative deprivation in individual utility, or as averages of altruistic well-being in the population, thus providing two alternative ways in which to incorporate interpersonal utility comparisons. Graphically, they are simply interpreted as averages of the product of individual utility and a rank-corrected aggregative weight.

7 Appendix

Proof of proposition 1.

Note first that Lorenz consistency implies and is implied by the S-concavity of $W(\mathbf{y})$ (see Dasgupta et al (1973)). Further, $W(\mathbf{y})$ is S-concave if and only if it does not decrease after a mean preserving progressive transfer. We consider first the sufficiency of conditions (9) and (10).
a)Sufficiency

i) If $\rho \geq 1,\ \epsilon \geq 0$ and $b > 0$, then $W(\mathbf{y})$ is S-concave.

Proof: Consider first a mean preserving marginal transfer of income $dy > 0$ from a rich (j) to a poor $(k > j)$ which does not affect the ranks. Then:

$$\begin{aligned} dW(\mathbf{y}) &= -g_j U'(y_j)dy \ + \ g_k U'(y_k)dy \\ &= [g_k U'(y_k)dy \ - \ g_j U'(y_j)] \geq 0 \end{aligned} \tag{1.A}$$

since $g_k \geq g_j > 0$ by $\rho \geq 1$ and $U'(y_k) \geq U'(y_j) \geq 0$ by $\epsilon \geq 0,\ b > 0$. Since $W(\mathbf{y})$ is continuous in y_i, $dW(\mathbf{y}) \geq 0$ even for a mean-preserving, inequality-reducing transfer which affects the ranks. Therefore, if $\rho \geq 1$, $\epsilon \geq 0$ and $b > 0$, $W(\mathbf{y})$ is S-concave and thus Lorenz consist.

ii) If $b > 0$, $\epsilon \geq 0$ and $\rho \geq 1$, then $W(\mathbf{y})$ is increasing in y_i. This is easily checked.

iii) If $g_i = i^\rho - (i-1)^\rho$, $\rho \geq 1$, then $W_{\rho,\epsilon}(\mathbf{y})$ obeys the Dalton population principle.

Proof: $W_{\rho,\epsilon}(\mathbf{y}^q)$ can be expressed as:

$$\begin{aligned} W_{\rho,\epsilon}(\mathbf{y}^q) &= \frac{\sum_{i=1}^n \sum_{j=1}^q [((i-1)q+j)^\rho \ - \ ((i-1)q+j-1)^\rho]\, U(y_i)}{\sum_{i=1}^n \sum_{j=1}^q [((i-1)q+j)^\rho \ - \ ((i-1)q+j-1)^\rho]} \\ &= \frac{\sum_{i=1}^n [(iq)^\rho \ - \ ((i-1)q)^\rho]\, U(y_i)}{(nq)^\rho} \\ &= W_{\rho,\epsilon}(\mathbf{y}) \end{aligned} \tag{1.B}$$

Thus, $W_{\rho,\epsilon}(\mathbf{y})$ obeys the Dalton population principle.

iv) If $U(y) = U_\epsilon(y)$, then $I_{\rho,\epsilon}(y)$ is a relative index of inequality.

Proof: This is easily checked: $I_{\rho,\epsilon}(\mathbf{y}) = I_{\rho,\epsilon}(\lambda\mathbf{y})$, for all $\lambda > 0$ and for all $\mathbf{y} \in \mathcal{R}_+^n$.

We now turn to the necessity of (9) and (10).

b) Necessity

i) If $W(\mathbf{y})$ obeys the principle of population, then by theorem 2 of Donaldson and Weymark (1980), we must have:

$$g_i = i^\rho - (i-1)^\rho. \tag{1.C}$$

ii) If $W(\mathbf{y})$ yields a relative inequality index, then it must be that $U(y) = U_\epsilon(y)$.

Proof: This is immediate from Atkinson (1970) and previous work (such as Pratt (1964)) by thinking of g_i as frequencies in a sum of $U(y_i)$.

iii) If $W(\mathbf{y})$ is increasing in y_i, then $b > 0$.
Proof: This is immediate since $g_i > 0$ and the derivative of $W(\mathbf{y})$ with respect to y_i is given by $bg_i y_i^{-\epsilon}$, and since $g_i y_i^{-\epsilon} > 0$ for all values of $y_i > 0$.

iv) If $W_{\rho,\epsilon}(\mathbf{y})$ is S-concave and increasing, then $\rho \geq 1$.
Proof: Assume that, on the contrary, $\rho < 1$; then, for $j < k$, $y_j \geq y_k$, and $g_j - g_k > 0$. Consider a marginal mean-preserving progressive income transfer from j to k. Then:

$$\begin{aligned} dW_{\rho,\epsilon}(\mathbf{y}) &= g_k U'_\epsilon(y_k) - g_j U'_\epsilon(y_j)) \\ &= (g_k - g_j)U'_\epsilon(y_k) + g_j(U'_\epsilon(y_k) - U'_\epsilon(y_j)) \\ &= (g_k - g_j)U'_\epsilon(y_k) + g_j U''_\epsilon(y^*)(y_k - y_j) \end{aligned} \tag{1.D}$$

where $y^* \in [y_k, y_j]$ since $U'(y)$ is continously differentiable for all $y > 0$. S-concavity of $W_{\rho,\epsilon}(\mathbf{y})$ requires that $dW(\mathbf{y}) \geq 0$. If $\epsilon \leq 0$, $U''_\epsilon(y^*) \geq 0$, and from (1.D) $dW_{\rho,\epsilon}(\mathbf{y}) \leq 0$, which means that $W_{\rho,\epsilon}(\mathbf{y})$ is not S-concave. If $\epsilon > 0$, then $U''_\epsilon(y^*) < 0$; by choosing y_j and y_k sufficiently close such that:

$$y_j - y_k < \frac{U'(y_k)(g_j - g_k)}{g_j(-U''(y^*))} \tag{1.E}$$

we have that $dW(\mathbf{y}) < 0$. Hence, whatever the value of ϵ, we must have $\rho \geq 1$ for the S-concavity of W(**y**).

v) If $W(\mathbf{y})$ is S-concave and increasing, then $\epsilon \geq 0$.
Proof: Assume on the contrary that $\epsilon < 0$. Consider a marginal mean-preserving progressive transfer from j to k, with $j < k$. Then, as before:

$$dW(\mathbf{y}) = g_j(U'_\epsilon(y_k) - U'_\epsilon(y_j)) + (g_k - g_j)U'_\epsilon(y_k) \tag{1.F}$$

For S-concavity of $W(\mathbf{y})$, we require $dW(\mathbf{y}) \geq 0$. However, by choosing k and j such that:

$$\frac{g_k - g_j}{g_j} < \frac{(U'(y_j) - U'(y_k))}{U'(y_k)} \tag{1.G}$$

we have that $dW(\mathbf{y}) < 0$. Such a choice is always posible. To see this, assume for simplicity that $k = j+1$. Then $y_k < y_j$, and since $\epsilon < 0$ and $b > 0$, $U'(y_j) > U'(y_k) > 0$. The right-hand side of (1.G) is therefore positive. There are two cases:

1) If $\rho \leq 1$, $g_k \leq g_j$, which by (1.G) leads to $dW(\mathbf{y}) < 0$.

2) If $\rho > 1$, then note that $\frac{d(g_{j+1}/g_j)}{dj} < 0$.

   To see this, note that $\frac{d(g_{j+1}/g_j)}{dj} < 0$ implies $g'_{j+1}g_j - g'_j g_{j+1} < 0$. If we replace g_j by $j^\rho - (j-1)^\rho$ and reorganise the last inequality, we find:

$$2(j-1)^{\rho-1}(j+1)^{\rho-1} < j^{\rho-1}\left((j-1)^{\rho-1} + (j+1)^{\rho-1}\right) < 0. \tag{1.H}$$

   This inequality can be seen to hold under three sets of values for ρ:

   a) If $1 < \rho < 2$, then $2j^{\rho-1} > (j-1)^{\rho-1} + (j+1)^{\rho-1}$, since the function $x^{\rho-1}$ is strictly concave. Suppose that $A = (j-1)^{\rho-1}$, $B = (j)^{\rho-1}$ and $C = (j+1)^{\rho-1}$. Then $2B > A + C$. Furthermore, if we suppose that $B = A + \eta_1 = C - \eta_2$, then $\eta_1 > \eta_2 > 0$ by the concavity of the function $x^{\rho-1}$. If we replace these results in (1.H), we find:

$$\begin{aligned} 2AC &< BA + BC \\ 2AC &< (C - \eta_2)A + (A + \eta_1)C \\ \eta_1 C &> \eta_2 A. \end{aligned} \tag{1.I}$$

   Since the last inequality holds, (1.H) must also hold.

   b) If $\rho > 2$, then $2j^{\rho-1} < (j-1)^{\rho-1} + (j+1)^{\rho-1}$, since the function $x^{\rho-1}$ is strictly convex. Hence, replacing $(j-1)^{\rho-1} + (j+1)^{\rho-1}$ by $2j^{\rho-1}$ on the left-hand side of (1.H), we find:

$$\begin{aligned} 2\left((j-1)(j+1)\right)^{\rho-1} &< 2(j^2)^{\rho-1} \\ (j^2)^{\rho-1} &> (j^2-1)^{\rho-1}. \end{aligned} \tag{1.J}$$

   Since the last inequality holds, (1.H) must also hold.

   c) It $\rho = 2$, the proof is trivial.

   Note also that:

$$\lim_{j\to\infty}\left(\frac{g_{j+1}}{g_j}\right) = 1 \tag{1.K}$$

   for all finite $\rho > 1$. Hence, it is enough to choose a sufficiently large value of j to obtain that the left-hand side of (1.G) be sufficiently close to zero, so that (1.G) holds and $dW(\mathbf{y}) < 0$. Therefore, for S-concavity of $W(\mathbf{y})$, we must have $\epsilon \geq 0$.

Proof of proposition 2.

The proof essentially flows from the proof of Proposition 1 in Duclos (2000). Using the definition of the generalised Lorenz curve of utilities and (17) and (18), we find that:

$$d_F(p_i) \;=\; \mu_U(F) - GL_F^U(p_i) - U(y(p_i))(1-p_i). \tag{2.A}$$

This yields the following ethically weighted average of relative deprivation:

$$\begin{aligned} D_\rho(F) &= \frac{1}{\rho}\int_0^1 d_F(p)k(p,\rho)dp \\ &= (\rho-1)\int_0^1 \{[\mu_U(F) - GL_F^U(p)](1-p)^{(\rho-2)} - [U(y(p))]\,(1-p)^{(\rho-1)}\}dp \end{aligned} \tag{2.B}$$

Proceeding by integration by parts for the term $U(y(p))(1-p)^{(\rho-1)}$ by integrating $U(y(p))$ to yield $GL_F^U(p)$ and differentiating $(1-p)^{(\rho-1)}$, we find that:

$$\begin{aligned} D_\rho(F) &= \frac{1}{\rho}\int_0^1 d_F(p)k(p,\rho)dp \\ &= \mu_U(F) - \int_0^1 GL_F^U(p)k(p,\rho)dp \\ &= \mu_U(F) - W_\rho(F) \end{aligned} \tag{2.C}$$

by equation (15). This demonstrates the proposition.

Proof of proposition 3.

For the proof, note first that by the definition of the altruistic individual well-being function, we have that:

$$A_F(p,\rho) = (1-p)^{\rho-1}U(y_F(p)) + \int_0^p (\rho-1)(1-q)^{\rho-2}U(y_F(q))dq. \tag{3.A}$$

Equation (3.A) says that $A_F(p,\rho)$ is a weighted average of p's egoistic utility function and of the utility of those that are poorer than him. The weight $(1-p)^{\rho-1}$ is the probability that the individual with rank p finds himself the least well-off in his comparison with the $\rho-1$ other individuals. The weight $(\rho-1)(1-q)^{\rho-2}$ is the density of an other individual (with utility $U(y(q))$) in the population being the least well-off in the comparison. Note that

$$(1-p)^{\rho-1} + \int_0^1 (\rho-1)(1-q)^{\rho-2}dq = 1. \tag{3.B}$$

Let then $\bar{U}_F(p,\rho)(p) = \int_0^p(\rho-1)(1-q)^{\rho-2}U(y_F(q))dq$. Using (3.A), this leads to:

$$\int_0^1 A_F(p,\rho)dp = \int_0^1 (1-p)^{\rho-1}U(y_F(p))dp + \int_0^1 \bar{U}_F(p,\rho)dp. \tag{3.C}$$

Integrating by parts the last term of (3.C):

$$\int_0^1 \bar{U}_F(p,\rho)dp = p\bar{U}_F(p,\rho)|_0^1 - \int_0^1 p(\rho-1)(1-p)^{\rho-2}U(y_F(p))dp \tag{3.D}$$

$$= \int_0^1 (\rho-1)(1-p)^{(\rho-1)}U(y_F(p))dp. \tag{3.E}$$

Combining (3.C), (3.E) and (14), we find the result of Proposition 3:

$$\int_0^1 A_F(p,\rho)dp = \int_0^1 \rho(1-p)^{\rho-1}U(y_F(p))dp = W_\rho(F). \quad \blacksquare \tag{3.F}$$

8 References

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Figure 1