E C O N O M I C S   B U L L E T I N

Negative results in the theory of games with lexicographic utilities

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Abstract

When players may have lexicographic utilities, there are: (i) extensive games having a non−empty set of equilibria but empty sets of sequentially rational, sequential and perfect equilibria; (ii) normal form games having a non−empty set of equilibria but an empty set of proper equilibria and no stable set of equilibria; and (iii) two extensive games having the same normal form representation and disjoint sets of sequential equilibria.

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1. Introduction

Fishburn (1972) shows that finite two-person zero-sum games may have no equilibrium if players' preferences are not represented by the standard von Neumann-Morgenstern utilities but by lexicographic (non-Archimedean) utilities; for lexicographic orders and lexicographic expected utility, see Hausner (1954), Thrall (1954) and Fishburn (1974).

Call "GT_1" the theory of non-cooperative games having von Neumann-Morgenstern utilities (the "standard" game theory) and "GT_2" the theory of non-cooperative games having lexicographic utilities. The significance of Fishburn's result lies in discovering the existence of a gap between GT_1 and GT_2, namely, that the basic existence theorem in GT_1, due to Nash (1951), according to which every game possesses at least one equilibrium in mixed strategies, does not hold in GT_2. This note tries to contribute to measure the separation between GT_1 and GT_2 by identifying four additional fundamental results in GT_1 that fail in GT_2.

The four results are the following. First, that every extensive game with perfect recall possesses a sequential equilibrium; see Kreps and Wilson (1982). Second, the sufficiency of the normal form principle, according to which all that is necessary to solve an extensive game is in its normal form representation; see Kohlberg and Mertens (1986, pp. 1010-1012) and Myerson (1991, pp. 50-51). Third, that every perfect equilibrium is a sequential equilibrium; see Selten's (1975) Proposition 5. And fourth, that a proper equilibrium of a reduced normal form is sequential in any extensive game with that normal form; see Proposition 0 in Kohlberg and Mertens (1986, p. 1009).

Section 2 shows that games with lexicographic utilities may have equilibria none of which satisfies the weak requirement of being sequentially rational (prescribing best replies at every information set). Section 3 shows that a game with lexicographic utilities having an empty set of sequentially rational equilibria may turn out to have some by inflating a player's information set and, simultaneously, preserving the normal form representation. Thus, if being sequentially rational is necessary for an equilibrium to be reasonable, two different games in GT_2 with the same normal form representation may have completely different sets of reasonable equilibria. Section 4 shows that, in an extensive game with lexicographic utilities, a perfect equilibrium need not be sequential and, moreover, that a proper equilibrium of its normal form representation need not induce a sequential equilibrium in the original extensive game. Section 5 discusses an expected utility version of the game with lexicographic utilities that yields the above results as well as ways of trying to solve the difficulties these results express.

2. Result I: non-empty set of equilibria but empty set of sequentially rational equilibria

Consider the extensive game with lexicographic utilities G_1 represented as Figure 1, where player 2's utilities are lexicographic (two-dimensional) utilities that are ordered by the lexicographic order $\geq^L$ such that $(x, y) \geq^L (x', y')$ if, and only if, $x > x'$ or ($x = x'$ and $y \geq y'$), with $(x, y) >^L (x', y')$ if, and only if, $(x, y) \geq^L (x', y')$ and not $(x', y') \geq^L (x, y)$. It then follows that $(1, 2) >^L (1, 1) >^L (0, 0)$. In G_1, while players 1 and 3 try to maximize standard expected utility, player 2 tries to maximize lexicographic expected utility.

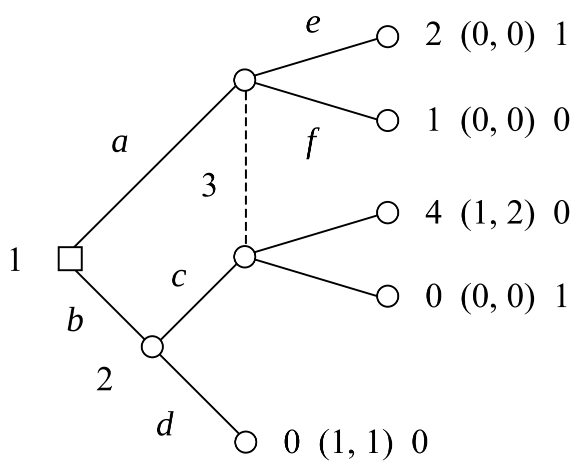

Fig. 1

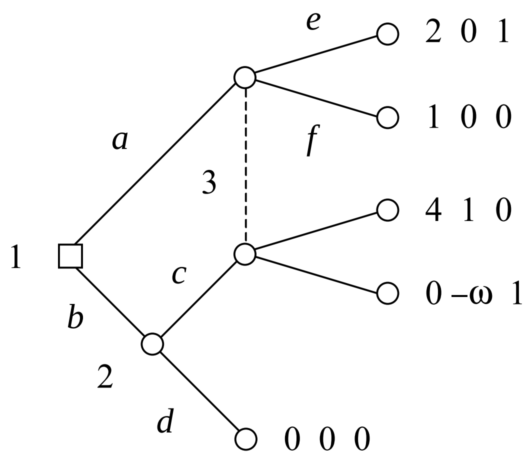

Fig. 2

Solution-concepts in GT_1 can be defined for G_1 by retaining the original formulation for players with standard utilities and by inserting the order $\geq^L$ for players with lexicographic utilities. For strategy profile σ and pure strategy $s \in \{a, b, c, d, e, f\}$, let σ_s denote the probability that σ assigns to s.

Remark 2.1. Every equilibrium σ of G_1 is such that $\sigma_a = \sigma_e = 1$ and $0 \leq \sigma_c \leq ½$.

It is easy to verify that (a, d, e) is the only pure-strategy equilibrium in G_1. Let σ be a strategy profile of G_1. Case 1: $0 \leq \sigma_a < 1$. Then choosing c yields player 2 expected lexicographic payoff $\sigma_b\sigma_e(1, 2)$, whereas choosing d yields $\sigma_b(1, 1)$. If $\sigma_e < 1$ then player 2's lexicographic best reply is d, in which case player 1's best reply is a, so σ with $\sigma_a < 1$ cannot be an equilibrium. If $\sigma_e = 1$ then player 2's lexicographic best reply is c, in which case player 3's best reply is f and σ with $\sigma_e = 1$ cannot be an equilibrium. As a result, no profile σ in G_1 such that $0 \leq \sigma_a < 1$ is an equilibrium. Case 2: $\sigma_a = 1$. Then player 3's best reply is e and to make a a best reply for player 1 given $\sigma_e = 1$, σ_c cannot be greater than ½.

Define strategy profile σ to be sequentially rational if, at every information set *h*, some probability distribution over the nodes of *h* makes the strategy that σ prescribes at *h* a best reply to what σ prescribes after *h*.

Remark 2.2. No equilibrium of G_1 is sequentially rational and, therefore, the set of sequential equilibria of G_1 is empty.

By Remark 2.1, in every equilibrium σ of G_1, $\sigma_e = 1$ and $0 \leq \sigma_c \leq ½$. As $(1, 2) >^L (1, 1)$, player 2's only lexicographic best reply to *e* at 2's information set consists of choosing *c* with probability 1, so G_1 has an empty set of sequentially rational equilibria and, hence, no sequential equilibrium. Consequently, Remarks 2.1 and 2.2 prove the following.

Proposition 2.3. *There are extensive games with lexicographic utilities having a non-empty set of equilibria but having both an empty set of sequentially rational equilibria and an empty set of sequential equilibria.*

3. Result II: insufficiency of the normal form

Consider now the game with lexicographic utilities G_3 represented as Figure 3. Game G_3 is obtained from G_1 by letting player 2 ignore whether player 1 has chosen *b* or not.

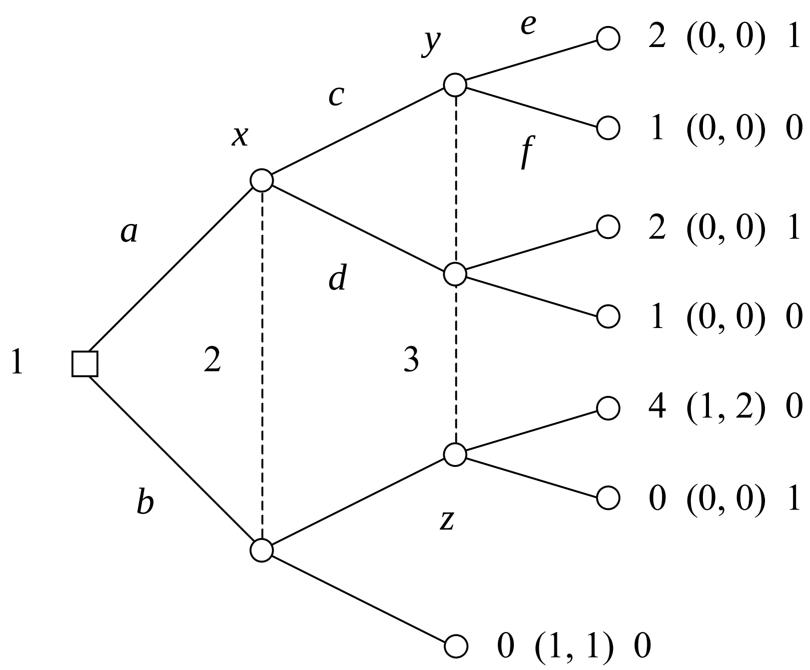

Fig. 3

Remark 3.1. G_3 has the same set of equilibria as G_1: strategy profile σ is an equilibrium of G_3 if, and only if, $\sigma_a = \sigma_e = 1$ and $0 \leq \sigma_c \leq ½$.

Remark 3.2. G_1 and G_3 have both the same (reduced) normal form representation.

Remark 3.3. Every equilibrium of G_3 is a sequential equilibrium.

In fact, every equilibrium σ of G_3 generates a sequential equilibrium with beliefs ascribing probability 1 to node *x*, probability σ_c to node *y* and probability 0 to node *z*. Accordingly, Remarks 2.2, 3.2 and 3.3 prove the following result.

Proposition 3.4. *There are extensive games G and G' with lexicographic utilities having the same (reduced) normal form representation with no equilibrium being sequential in both G and G'.*

By Proposition 3.4, the respective extensive forms of G_1 and G_3 cannot be regarded as different presentations of the same decision problem: the fact that in G_1, in contrast to G_3, player 2 realizes, when called upon to play, whether player 1 has played his unique equilibrium strategy *a* is an important difference and not a mere "presentation effect". It is having this information that causes the non-existence of sequential equilibria in G_1. As a result, the example consisting of G_1 and G_3 define a candidate to disprove the sufficiency of the normal form principle in GT_2.

4. Result III: neither perfect nor proper implies sequential

Since each player has one information set in G_1, Selten's (1975, p. 38) perfect equilibria can be obtained from the normal form representation N_1 of G_1 by considering sequences of completely mixed strategy profiles. Specifically, for $s \in \{a, b, c, d, e, f\}$, let ε_s (with $0 < \varepsilon_s < ½$) be the minimum probability with which *s* has to be played in N_1. Then player 2's only lexicographic best reply consists of playing *d* with probability $1 - \varepsilon_c$. Given this, player 1's best reply is to play *a* with probability $1 - \varepsilon_b$. And given this, player 3's best reply is to play *e* with probability $1 - \varepsilon_f$. Thus, the only equilibrium that can be obtained as the limit of a sequence of (perturbed) completely mixed strategies in which players play best replies is (*a*, *d*, *e*). This means that G_1 and N_1 have only one perfect equilibrium. Moreover, given that every player has exactly two pure strategies in N_1, the set of Myerson's (1978) proper equilibria of N_1 coincides with the set of perfect equilibria. The following result summarizes these conclusions.

Remark 4.1. Strategy profile (a, d, e) is: (i) the only perfect equilibrium of G_1; (ii) the only perfect equilibrium of N_1; and (iii) the only proper equilibrium of N_1.

Proposition 4.2. *A perfect equilibrium of an extensive game with lexicographic utilities need not be a sequential equilibrium of the game.*

Proposition 4.3. *A proper equilibrium of the normal form of an extensive game with lexicographic utilities need not be sequential in all extensive games with that normal form.*

Proposition 4.2 follows from Remarks 2.2 and 4.1(i). By Proposition 4.3, which follows from Remarks 2.2 and 4.1(ii), in GT_2, properness in the normal form does not embody sequential rationality in the extensive form: an extensive game may have an empty set of sequential equilibria but its normal form representation may have a non-empty set of proper equilibria. Note that in every perturbed game of N_1, player 2's only best reply is d; if d is played with sufficiently high probability, player 1's only best reply is a; and if a is played with sufficiently high probability, player 3's only best reply is e. Hence, {(a, d, e)} is a stable set of equilibria of N_1. These three results seem to indicate that, in the lexicographic case, perturbing a game to find a solution for the game itself may not be appropriate: in G_1, the solution-concepts of perfect equilibria, proper equilibria and stable sets of equilibria fail to recognize the unreasonability (if sequential rationality is necessary for being reasonable) of all the equilibria of G_1.

5. Comments

For comparison, the referee suggests analyzing an expected utility version of G_1. Consider, for instance, the extensive game G_2 represented as Figure 2, where $\omega > 1$ is an arbitrarily large real number. This game has two types of equilibria. Every equilibrium σ of the first type is such that $\sigma_a = \sigma_e = 1$ and $0 \leq \sigma_c \leq ½$ and the only equilibrium $\tau = (\tau_a, \tau_c, \tau_e)$ of the second type satisfies $\tau_a = (1 + 2\omega) / (1 + 6\omega)$, $\tau_c = (1 + 2\omega) / 4\omega$ and $\tau_e = \omega / (1 + \omega)$. Whereas no equilibrium of the first type is sequential, τ is a sequential equilibrium. When ω is infinitely large, every equilibrium of the first type remains an equilibrium but τ approaches (⅓, ½, 1), which is not an equilibrium. In addition, by the same arguments as in Section 4, (a, d, e) is the only perfect equilibrium of G_2 and the only proper equilibrium of its normal form representation. Therefore, when ω is unbounded, G_2 has a non-empty set of perfect equilibria but empty sets of sequential and sequentially rational equilibria.

Since Fishburn's (1972) equilibrium existence problem stems from the introduction of non-standard utilities, Skala (1974, pp. 77-79; 1975, pp. 111-114) suggests as a solution to allow non-standard probability weights; see Skala's (1974, p. 79) Theorem 13 or Skala's (1975, pp. 113-114) Theorem 5. Skala's solution amounts in G_2 to consider τ itself an admissible strategy profile, so that player 3, for instance, can ascribe probability $\omega / (1 + \omega)$ to *e*, where ω is an infinitely large number.

This extension is analogous to the mixed strategy extension applied to standard games without pure strategy equilibria. Rajan (1998) presents an extension of GT_1 when players can believe that opponents may tremble with infinitesimal probability (with an infinitesimal number represented by a sequence of real numbers tending to zero); see also Hammond (1999). Skala's approach seems then to resolve the difficulties of games like G_2: if unbounded payoffs are allowed then allow as well unbounded probabilities (which is what makes it possible for player 2 to be indifferent at his information set). It is nonetheless not obvious that such an approach is equally helpful to deal with the difficulties arising in games like G_1, because no number (real or infinitesimal) ascribed as a probability to either *e* or *f* can make player 2 indifferent between *c* and *d* at his information set.

Being the introduction of lexicographic utilities the source of the problem, the referee suggests introducing the lexicographic beliefs proposed by Blume, Branderburger and Dekel (1991), who use them to characterize perfect and proper equilibria. Lexicographic beliefs are vectors of probability distributions that are interpreted as the players' first-order, second-order and higher order conjectures as to how the opponents will play and are used lexicographically to determine best replies. So, for instance, player 2 in G_1 may entertain the lexicographic belief in which the first-order belief is that $\sigma_a = \sigma_e = 1$ will be played and the second-order belief that $\tau_a = \tau_e < 1$ will be played. In this case, playing *d* is a best reply to both the first-order and the second-order belief.

The apparent initial unreasonability of equilibrium (*a*, *d*, *e*) may be removed with the introduction of lexicographic beliefs for the player with lexicographic utilities. The problem is that such beliefs do not make (*a*, *d*, *e*) sequentially rational, if the beliefs that must count are those consistent with the reaching of player 2's information set and with the strategy specification for the players that may play after 2's information set (in the case at hand, player 3). This maybe suggests that sequential rationality would have to be discarded as a reasonable requirement in GT_2 (or completely redefined). But if one is willing to reject this principle, why not reconsider the notion of rationality attributed to players in GT_2? Rather than imposing a concept of rationality in every conceivable game situation, why not adapt the concept of rationality to the specific game situation? Specifically, in G_1, when called upon to play, player 2 must decide whether or not running the risk of obtaining (0, 0) just to achieve the infinitesimal improvement from (1, 1), which is what he can ensure himself by choosing *d*, to (1, 2). This situation would then point to a notion of cautious rationality that justifies playing *d* regardless of the belief concerning player 3's choice held by player 2.

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