E C O N O M I C S   B U L L E T I N

Voting Power Under Uniform Representation

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Abstract

We use the Banzhaf power index to compare the voting power induced by a simple popular vote and that induced by a uniform electoral college system in which the nation's population is partitioned into "states" of equal size for election purposes. While the adoption of uniform representation would remedy the well−known inequality of voting power that is inherent to the Electoral College system as practiced, we show that substantial negative effects remain. We measure the magnitude of these negative effects with the help of Stirling's Formula.

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1. Introduction

The Electoral College presidential election system was devised by the United States founding fathers to serve as a compromise between an election by Congress and an election by nationwide popular vote. In this system, the voters of each state and the District of Columbia elect their allotted members of the Electoral College, who in turn elect the president. (Each state has one elector for each of its Representatives and Senators. The District of Columbia is allotted three electors.) It is well known than that the Electoral College falls far short of providing United States citizens with equal power in determining their president. For instance, both Banzhaf (1968) and Owen (1975) argue that Californians control over three times the power to determine their nation's leader than do citizens of the District of Columbia. One hypothesis that naturally emerges is that variability in state populations is the sole source of "ill" effects on voting power. We test this hypothesis by examining voting power in an electoral college with uniform representation. To be precise, we consider the hypothetical effects of forming voting districts ("states") of equal populations so that each would have an equal number of electors and every citizen would be endowed with equal voting power. Our results reveal that even though a mandate of uniform representation would equalize voting power across citizens, substantial deleterious effects on absolute voting power would remain.

Previous research in random graph theory (see Lemma 6.1 of Friedgut and Kalai (1996)) has demonstrated that a simple popular vote represents the "gold standard" in regards to maximizing the average voting power of citizens in a random voting model. That is, no form of "coalitional voting" can yield higher average voting power than a popular vote. (Gelman, Katz, and Tuerlinckx (forthcoming) also derives this conclusion, but does so in the direct context of a voting model.) While this result does establish the voting power dominance of a popular vote, it does not indicate the magnitude of this domination. If political concerns other than just relative and absolute voting power merit attention, it may still be important to identify the degree to which voting power is compromised at various population levels. We use Stirling's formula to construct estimates of the voting power losses realized by a uniform electoral college. Our analysis also provides well-defined bounds on the errors of these estimates.

Before turning to the main body of this short paper, we wish to acknowledge that there is no universally accepted measure of voting power, even within the relatively narrow confines of elementary game theoretic analysis. (See, for instance, early classics such as Shapley and Shubik (1954), Riker (1963), and Banzhaf (1965).) We happen to adopt the approach advocated in Banzhaf (1965) and (1968), the latter of which states:

> "In any voting situation it is possible to consider all of the possible ways in which the different voters could vote; i.e., to imagine all possible voting combinations. One then asks in how many of these voting combinations can each voter affect the outcome by changing his vote. Since, a priori, all voting combinations are equally likely and therefore equally significant, the number of combinations in which each voter can change the outcome by changing his vote serves as the measure of his voting power."

We do not, however, mean to suggest that this approach is the definitively best measurement of voting power. Indeed, the measurement is surely flawed to whatever extent various voting combinations may not be equally likely in specific applications. Even so, for the purposes of assessing election procedure performance, it is informative to examine this performance throughout every conceivable contingency.

2. Stirling's Formula and Voting Power

We use the Banzhaf index to measure individual voting power in two-candidate elections. Each voter's power will thus be measured as the fraction of all conceivable voting profiles for which the individual's vote is pivotal in its effect on the election's outcome. Our goal is to establish a transparent representation of the voting power differential between a popular vote and uniform electoral college systems. A direct calculation of this differential, as we shall see in the proof of our main theorem, is interlaced with factorials and is not a particularly revealing structure for considering arbitrary population levels. *Stirling's Formula* proves to be very useful in this regard, particularly the formulation offered by Robbins (1955), which states that for all positive integers n, it follows that

$$n! = \left(e^{-n} n^n \sqrt{2\pi n}\right) e^{r_n} \qquad (1)$$

where r_n satisfies

$$\frac{1}{12n+1} < r_n < \frac{1}{12n} \qquad (2)$$

We will show that using equation (1) to replace the factorials embedded in voting power indices enables an algebraic reduction that is remarkably transparent, while the inequalities of (2) provide well-defined bounds on estimation error.

In stating our main result below, it is helpful to define for all positive integers z the value $h(z)=z/2$ if z is even and $h(z)=(z-1)/2$ if z is odd. Thus $h(z)$ is the integer part of one half z.

THEOREM If a nation is partitioned into $M>1$ "states" each of size $N>1$, then the ratio of voting power in a popular vote to that in the corresponding electoral college system is equal to $\frac{\sqrt{\pi}}{\sqrt{2+\frac{1}{m}+\frac{1}{n}}} e^{q_{m,n}}$, where $m=h(M)$, $n=h(N)$, and $q_{m,n}=r_{4mn+2m+2n}-2r_{2mn+m+n}-(r_{2m}+r_{2n})+2(r_m+r_n)$. Furthermore, inequalities (9) and (10), found in our proof below, place bounds on $q_{m,n}$, which in turn imply that $q_{m,n}$ is positive and converges to 0 as m and n go to infinity.

Proof: First note that in a field of $2X+1$ voters, there are 2 different ways a given voter can cast a vote and $(2X)!/(X!X!)$ different ways the remaining voters can be evenly split between the two candidates. It follows that in a popular vote, the number of profiles in which a given voter is pivotal can be expressed by

$$2\frac{(2X)!}{X!X!} \tag{3}$$

Next note that Stirling's formula, as expressed in (1) and (2) above, can be used to establish that

$$\frac{(2X)!}{X!X!}=\frac{2^{2X}}{\sqrt{\pi X}}e^{r_{2X}-2r_X} \tag{4}$$

Let us now assume the nation's population is partitioned into M=2m+1≥3 voting districts ("states"), each of size N=2n+1≥3, i.e., an odd number of three or more states with an odd number of three or more citizens. (The cases of M or N being even also follows from the logic we present, as we establish at the end of this proof.) The number of voting profiles in which a given citizen is pivotal under a popular vote can thus be expressed as:

$$\frac{2(4mn+2m+2n)!}{(2mn+m+n)!(2mn+m+n)!}=\frac{2^{4mn+2m+2n+1}}{\sqrt{\pi(2mn+m+n)}}e^{r_{4mn+2m+2n}-2r_{2mn+m+n}} \tag{5}$$

If an electoral college system is adopted, then the number of voting profiles in which a citizen is pivotal depends on that citizen's state being pivotal *and* the citizen being pivotal within this state. Thus if a nation is partitioned as described above, there are $\frac{(2m)!}{m!m!}$ different ways states other than that of the given citizen can be evenly split. Each of these states has exactly 2^{2n} of its citizens' voting profiles (half of those possible) which yield a specific winning candidate. It follows that the number of profiles in which a given citizen is pivotal can be represented by:

$$\frac{(2m)!}{m!m!}(2^{2n})^{2m}\frac{2(2n)!}{n!n!}=\frac{2^{4mn+2m+2n+1}}{\pi\sqrt{mn}}e^{r_{2m}+r_{2n}-2(r_m+r_n)} \tag{6}$$

The ratio of individual voting power under a popular vote and voting power under a uniform electoral college can thus be expressed as the ratio of (5) divided by (6), which is equal to:

$$\frac{\sqrt{\pi}}{\sqrt{2+\frac{1}{m}+\frac{1}{n}}}e^{r_{4mn+2m+2n}-2r_{2mn+m+n}-(r_{2m}+r_{2n})+2(r_m+r_n)} \tag{7}$$

The arguments above also apply to the cases of M or N being even by assuming that every tie is broken by a "coin flip" and noting that

$$2\frac{(2N-1)!}{N!(N-1)!}=\frac{(2N)!}{N!N!} \tag{8}$$

Finally, note that inequality (2) and definition of $q_{m,n}$ implies the following two inequalities

$$\frac{1}{12(4mn+2m+2n+1)}-\frac{2}{12(2mn+m+n)}-\frac{1}{12(2m)}-\frac{1}{12(2n)}+\frac{2}{12m+1}+\frac{2}{12n+1}<q_{m,n} \qquad (9)$$

$$q_{m,n}<\frac{1}{12(4mn+2m+2n)}-\frac{2}{12(2mn+m+n+1)}-\frac{1}{12(2m)+1}-\frac{1}{12(2n)+1}+\frac{2}{12m}+\frac{2}{12n} \qquad (10)$$

Our proof concludes by noting that the left-hand side of (9) is strictly positive and that both the left-hand side of (9) and the right-hand side of (10) converge to 0 as *m* and *n* go to infinity. QED

The central intuition behind our result can be seen by considering equations (5) and (6). The left-hand sides of these equalities are derived by directly calculating voting power under a popular vote and a uniform electoral college respectively. The right-hand sides are then constructed by using Stirling's formula to replace all factorials in the left-hand side expressions. Note that $2^{4mn+2m+2n+1}$ represents the total number of voting profiles possible in a national population of $4mn+2m+2n+1$ voters. Thus inequalities (5) and (6) can be interpreted as identifying the proportion of all profiles in which a given voter is pivotal in the respective voting system. Our conclusion follows by forming the ratio of these two proportions. The bounds that we establish for $q_{m,n}$ follow directly from (2).

Our Theorem demonstrates that while the formation of equal population voting districts (states) would ensure an equal distribution of voting power under the Electoral College system, it would not restore the voting power citizens would have enjoyed under a popular vote. In particular, dividing the current United States population equally between 51 voting districts (to arbitrarily reflect the current number of states plus the District of Columbia), it follows 1/n is negligible and 1/m=0.04. Application of our Theorem and inequalities (9) and (10) thus implies that the individual voting power supported by a popular vote is approximately 24.72% higher than that which would be generated by an electoral college system with uniform representation.

Finally, we note that the voting power enjoyed by citizens will presumably also affect, to at least some degree, their incentives to exercise their voting rights. In particular, the relatively large degree of voting power that a popular vote bestows upon citizens suggests a tendency to induce larger turnouts at the polls than would a uniform electoral college system.

References

Banzhaf, J. F. (1965) "Weighted Voting Doesn't Work: A Mathematical Analysis" *Rutgers Law Review* **19**, 317-343.

Banzhaf, J. F. (1968) "One Man, 3,312 Votes: A Mathematical Analysis of the Electoral College" *Villanova Law Review* **13**, 304-332.

Friedgut, E., and G. Kalai (1996) "Every Monotone Graph Property has a Sharp Threshold" *Proceedings of the American Mathematical Society* **124**, 2993-3002.

Gelman, A., J. N. Katz, and F. Tuerlinckx (forthcoming) "The Mathematics and Statistics of Voting Power" *Statistical Science*.

Owen, G. (1975) "Evaluation of a Presidential Election Game" *American Political Science Review* **69**, 947-953.

Riker, W. H. (1964) "Some Ambiguities in the Notion of Power" *American Political Science Review* **58**, 341-349.

Robbins, H. (1955) "A Remark on Stirling's Formula" *American Mathematical Monthly* **62**, 26-29.

Shapley, L. S., and M. Shubik (1954) "A Method for Evaluating the Distribution of Power in a Committee System" *American Political Science Review* **48**, 787-792.