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A necessary and sufficient condition for Wilson's impossibility theorem with strict non−imposition

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Abstract

Wilson's impossibility theorem (Wilson(1972)) about Arrovian social welfare functions (Arrow(1963)) states that there exists a dictator or an inverse−dictator for any non−null social welfare function which satisfies the conditions of unrestricted domain, non−imposition and independence of irrelevant alternatives (IIA). Among these conditions IIA is very strong and controversial. We will show that, under the condition of strict non−imposition which is stronger than non−imposition, IIA can be replaced by weaker condition. We call this condition "monotonicity". We will also show that under strict non−imposition it is necessary and sufficient condition for Wilson's theorem, that is, it is equivalent to dictatorship or inverse−dictatorship of Arrovian social welfare functions under unrestricted domain and strict non−imposition.

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1 Introduction

Wilson's impossibility theorem (Wilson (1972), or Binmore (1976)) about Arrovian social welfare functions (Arrow (1963)) states that there exists a dictator or an inverse-dictator for any non-null social welfare function (SWF) which satisfies the conditions of unrestricted domain, non-imposition and independence of irrelevant alternatives (IIA). Arrovian social welfare functions (SWFs) are collective choice rules which are complete and transitive. They are null if the social preferences generated by them are indifferent over all pairs of alternatives. Among above conditions IIA is very strong and controversial. We will show that, under the condition of strict non-imposition which is stronger than non-imposition, IIA can be replaced by a weaker condition. We call this condition *monotonicity* . We will also show that under strict non-imposition monotonicity is necessary and sufficient condition for Wilson's theorem, that is, it is equivalent to the existence of one of dictator and inverse-dictator for Arrovian social welfare functions under unrestricted domain and strict non-imposition.

Monotonicity is a two-fold condition. One of the following conditions (1) and (2) is satisfied.

(1) If for a pair of alternatives x and y at some profile p:
   (i) individuals in a group G prefer x to y,
   (ii) individuals in a group G' are indifferent between them,
   (iii) the other individuals prefer y to x,
   and the society prefers x to y, then at another profile where individuals in G and G' prefer x to y, the society must prefer x to y.
(2) If for a pair of alternatives x and y at some profile p:
   (i) individuals in a group G prefer x to y,
   (ii) individuals in a group G' are indifferent between them,
   (iii) the other individuals prefer y to x,
   and the society prefers y to x, then at another profile where individuals in G and G' prefer x to y, the society must prefer y to x.

We call the former *the normal form* and the latter *the inverse form* of monotonicity.

Strict non-imposition for a SWF means that for any pair of alternatives $x, y \in A$ there exists at least one profile of individual preferences such that the society prefers x to y. [1]

In the next section we present notations, terminologies and some preliminary results. In that section we will show that under strict non-imposition any SWF satisfies weak Pareto optimality or inverse weak Pareto optimality if it satisfies monotonicity. In Section 3 we will prove Wilson's impossibility theorem under strict non-imposition using monotonicity instead of IIA. In Section 4 we will show that this condition is equivalent to the existence of one of dictator and inverse-dictator for Arrovian social welfare functions under unrestricted domain and strict non-imposition. In the Appendix we will show that IIA with weak Pareto optimality (respectively, inverse weak Pareto optimality) implies the normal form (respectively, inverse form) of monotonicity.

[1] On the other hand, *non-imposition* for a SWF means that for any pair of alternatives $x, y \in A$ there exists at least one profile such that the society prefers x to y, or is indifferent between them.

2 Notations, definitions and preliminary results

A is the set of alternatives. The number of alternatives is larger than 2. $N = \{1, 2, \ldots, n\}$ is the finite set of individuals with $n \geqq 2$. Each individual i is endowed with a weak ordering R_i of A. A weak ordering is a complete and transitive binary relation. The strict preference and the indifference of individual i are denoted by P_i and I_i. Let $\mathscr{R}$ denote the set of all weak orderings of A. A profile p is a function mapping N into the set $\mathscr{R}^n$ of all logically conceivable profiles. For each i p assigns a weak ordering R_i, p' assigns a weak ordering R'_i and so on.

A collective choice rule (CCR) is a mapping of $\mathscr{R}^n$ into the set of social preferences over A. If a CCR generates a complete and transitive social preference, then it is an *Arrovian social welfare function(SWF)*. At a profile p the social preference is denoted by R, at a profile p' it is denoted by R' and so on. The strict preference and the indifference of R are denoted by P and I and so on.

The conditions for SWFs to satisfy other than completeness and transitivity are as follows.

Unrestricted domain The domain of any SWF is the set of all logically conceivable profiles, $\mathscr{R}^n$.

Strict non-imposition If for some pair of alternatives $x, y \in A$ the social preference is always xRy irrespective of the preferences of individuals, the SWF is weakly imposed. We assume that for any pair of alternatives any SWF is not weakly imposed. Strict non-imposition for a SWF implies that for any pair of alternatives $x, y \in A$ there exists at least one profile such that the social preference is xPy.

We do not require the condition of independence of irrelevant alternatives (IIA). Instead we assume the following condition.

Monotonicity Any social welfare function satisfies one of the following conditions.

(Normal form) Suppose that at some profile p, for any pair of alternatives $x, y \in A$:

(1) individuals in a group G: xP_iy,
(2) individuals in a group G': xI_iy,
(3) individuals in $N - (G \cup G')$: yP_ix,

and the social preference is xPy. At another profile p' if xP'_iy for individuals in G and G', then the social preference must be $xP'y$.

(Inverse form) Suppose that at some profile p, for any pair of alternatives $x, y \in A$:

(1) individuals in a group G: xP_iy,
(2) individuals in a group G': xI_iy,
(3) individuals in $N - (G \cup G')$: yP_ix,

and the social preference is yPx. At another profile p' if xP'_iy for individuals in G and G', then social preference must be $yP'x$.

Weak Pareto optimality For any pair of alternatives $x, y \in A$, when all individuals prefer x to y (xP_iy for all $i \in N$), the social preference must be xPy.

Inverse weak Pareto optimality For any pair of alternatives $x, y \in A$, when all individuals prefer x to y (xP_iy for all $i \in N$), the social preference must be yPx.

We do not assume weak Pareto optimality or inverse weak Pareto optimality. They are derived from strict non-imposition and monotonicity.

We define some other terminologies.

Almost decisive If for any pair of alternatives $x, y \in A$, when individuals in a group G prefer x to y and the other individuals prefer y to x, the social preference is xPy, then G is *almost decisive* over $\{x, y\}$.

We call a set of individuals which is almost decisive over any pair of alternatives an *almost decisive set*.

Decisive If for any pair of alternatives $x, y \in A$, when individuals in a group G prefer x to y, the social preference is xPy, then G is *decisive* over $\{x, y\}$.

Since in the definition of *decisive* it is not assumed that individuals other than those in G prefer y to x, if G is decisive over $\{x, y\}$, then it is almost decisive over $\{x, y\}$.

Using *decisive*, *dictator* is described as follows.

Dictator The dictator is an individual who is decisive over all pairs of alternatives.

That is, if G is decisive over all pairs of alternatives, and G consists of only one individual, he is the dictator.

Inversely almost decisive If for any pair of alternatives $x, y \in A$, when individuals in a group G prefer x to y and the other individuals prefer y to x, the social preference is yPx, then G is *inversely almost decisive* over $\{x, y\}$.

We call a set of individuals which is inversely almost decisive over any pair of alternatives an *inversely almost decisive set*.

Inversely decisive If for any pair of alternatives $x, y \in A$, when individuals in a group G prefer x to y, the social preference is yPx, then G is *inversely decisive* over $\{x, y\}$.

Using *inversely decisive*, *inverse-dictator* is described as follows.

Inverse-dictator The inverse-dictator is an individual who is inversely decisive over all pairs of alternatives.

That is, if G is inversely decisive over all pairs of alternatives, and G consists of only one individual, he is the inverse-dictator.

We can show the following lemma which is similar to Proposition 1 in Malawski and Zhou (1994).

Lemma 1 *For any SWF which satisfies unrestricted domain and strict non-imposition:*

(1) If a SWF satisfies the normal form of monotonicity, we obtain weak Pareto optimality, that is, for any pair of alternatives $x, y \in A$ if xP_iy for all $i \in N$ we have xPy.

(2) If a SWF satisfies the inverse form of monotonicity, we obtain inverse weak Pareto optimality, that is, for any pair of alternatives $x, y \in A$ if xP_iy for all $i \in N$ we have yPx.

PROOF.

(1) By strict non-imposition for any pair of alternatives $x, y \in A$ there is a profile p^1 where the social preference is xP^1y. Then, if the SWF satisfies the normal form of monotonicity, it

implies that at any profile p if xP_iy for all $i \in N$ we have xPy. Therefore, we obtain weak Pareto optimality.

(2) By strict non-imposition for any pair of alternatives $x, y \in A$ there is a profile p^1 where the social preference is yP^1x. Then, if the SWFs satisfies the inverse form of monotonicity, it implies that at any profile p if xP_iy for all $i \in N$ we have yPx. Therefore, we obtain inverse weak Pareto optimality. □

3 Wilson's impossibility theorem

In this section we prove the following theorem.

Theorem 1 *There exists one of dictator and inverse-dictator for any SWF which satisfies unrestricted domain, strict non-imposition and monotonicity.*

If a SWF satisfies the normal form of monotonicity, we obtain dictatorship, and if it satisfies its inverse form, we obtain inverse-dictatorship. The proofs of two cases are parallel. So we present a proof of only the inverse-dictatorship case. First we show the following standard lemma using the inverse form of monotonicity instead of IIA.

Lemma 2 *Suppose that a SWF satisfies the conditions of unrestricted domain, strict non-imposition and the inverse form of monotonicity. If, for some pair of alternatives $x, y \in A$, a group G is inversely almost decisive over $\{x, y\}$, then it is inversely decisive over all pairs of alternatives.*

PROOF. Let z be an alternative other than x and y. Consider the following profile p:

(1) individuals in G: xP_iyP_iz,
(2) individuals in $N - G$: yP_izP_ix.

Note that we can assume the existence of such a profile because we assume unrestricted domain for social welfare functions.

Since G is inversely almost decisive over $\{x, y\}$, we have yPx. By (2) of Lemma 1 we have inverse weak Pareto optimality. It implies zPy because all individuals prefer y to z. Then, by transitivity we obtain zPx. Since individuals in G prefer x to z and individuals in $N - G$ prefer z to x, inverse form of monotonicity implies that G is inversely decisive over $\{x, z\}$.

Let w be an alternative other than x and y. Consider the following profile p':

(1) individuals in G: $wP'_ixP'_iy$,
(2) individuals in $N - G$: $yP'_iwP'_ix$.

Since G is inversely almost decisive over $\{x, y\}$, we have $yP'x$. Inverse weak Pareto optimality implies $xP'w$ because all individuals prefer w to x. By transitivity we obtain $yP'w$. Since individuals in G prefer w to y and individuals in $N - G$ prefer y to w, inverse form of monotonicity implies that G is inversely decisive over $\{w, y\}$.

Next consider the following profile p'':

(1) individuals in G: $wP''_i xP''_i z$,
(2) individuals in $N - G$: $zP''_i wP''_i x$.

Since G is inversely decisive over $\{x, z\}$, we have $zP''x$. Inverse weak Pareto optimality implies $xP''w$ because all individuals prefer w to x. By transitivity we obtain $zP''w$. Since individuals in G prefer w to z and individuals in $N - G$ prefer z to w, inverse form of monotonicity implies that G is inversely decisive over $\{w, z\}$.

Because w and z are arbitrary, repeatedly applying this logic the proof of this lemma will be completed. □

By inverse weak Pareto optimality the set of all individuals N is inversely decisive over all pairs of alternatives. It means that there exists at least one set of individuals which is inversely almost decisive over some pair of alternatives, and since the number of individuals is finite, there exists a minimum set which is inversely almost decisive over some pair of alternatives. We call such a set a *minimum inversely almost decisive set*. *Minimum* means that the number of individuals included in the set is minimum among all inversely almost decisive sets. Then, we can show the following results.

Lemma 3 *Suppose that a SWF satisfies the conditions of unrestricted domain, strict non-imposition and the inverse form of monotonicity.*

(1) If there are two inversely almost decisive sets, the set which is the intersection of these two sets is also an inversely almost decisive set.
(2) We can not have multiple disjoint inversely almost decisive sets.
(3) We can not have multiple different minimum inversely almost decisive sets.

PROOF.

(1) Denote two inversely almost decisive sets by G_1 and G_2. By Lemma 2 G_1 and G_2 are inversely decisive over all pairs of alternatives. Select three alternatives x, y and z, and consider the following profile p:
 (i) individuals in $G_1 - G_2$: $xP_i yP_i z$,
 (ii) individuals in $G_1 \cap G_2$: $zP_i xP_i y$,
 (iii) individuals in $G_2 - G_1$: $yP_i zP_i x$,
 (iv) individuals in $N - (G_1 \cup G_2)$: $yP_i xP_i z$.
 Since G_1 is an inversely almost decisive set, we have yPx. Similarly, since G_2 is an inversely almost decisive set, we have xPz. Then, by transitivity we have yPz. Since only individuals in $G_1 \cap G_2$ prefer z to y and all other individuals prefer y to z, by monotonicity $G_1 \cap G_2$ is inversely decisive over $\{z, y\}$, and so it is inversely almost decisive over $\{z, y\}$.
(2) By Lemma 2 all inversely almost decisive sets are inversely decisive over all pairs of alternatives. Suppose that G and G' are two disjoint such sets. If all individuals in G prefer x to y, the social preference must be yPx. If, at the same time, all individuals in G' prefer y to x, the social preference must be xPy. It is a contradiction.
(3) Suppose that there are two different minimum inversely almost decisive sets, and denote them by G and G'. Then, from (1) of this lemma the intersection of G and G' is also an inversely almost decisive set. Therefore, neither G nor G' can not be the minimum inversely almost decisive set. □

This lemma means that the minimum inversely almost decisive set is unique. Then, we obtain the following conclusion.

Lemma 4 *Suppose that a SWF satisfies the conditions of unrestricted domain, strict non-imposition and the inverse form of monotonicity. The minimum inversely almost decisive set consists of only one individual.*

PROOF. Denote the minimum inversely almost decisive set by G, and assume that it includes more than one individual. Consider the following profile p:

(1) one individual in G (denoted by j): zP_jxP_jy,
(2) individuals in G other than j ($G - \{j\}$): xP_iyP_iz,
(3) individuals in $N - G$: yP_izP_ix.

z is an alternative other than x and y. Since G is an inversely almost decisive set, we have yPx. If the social preference is xRz, by transitivity we have yPz. Then, since only individual j prefers z to y and all other individuals prefer y to z, inverse form of monotonicity and Lemma 2 imply that he is the inverse-dictator. Thus, if there is no inverse dictator, the social preference must be zPx. Then, since individuals in $G - \{j\}$ prefer x to z and all other individuals prefer z to x, by monotonicity it is an inversely almost decisive set. It contradicts the assumption that G is the minimum inversely almost decisive set. Therefore, G consists of only one individual. He is the inverse-dictator. □

PROOF. [Proof of Theorem 1]

Theorem 1 is obtained from Lemmas 2, 3, and 4.

4 Equivalence of dictatorship or inverse-dictatorship and monotonicity

In this section we show the following result.

Theorem 2 *Monotonicity is equivalent to the existence of one of dictator and inverse-dictator for SWFs under unrestricted domain and strict non-imposition.*

PROOF. Theorem 1 has shown that monotonicity implies dictatorship or inverse-dictatorship of SWFs so that only the converse needs to be proved.

Assume that a dictatorial or inversely dictatorial SWF does not satisfy monotonicity. Then,

(1) **Dictatorial SWF case**: There is a case where for some pair of alternatives $x, y \in A$ we have two profiles p and p' such that
   (i) The social preference is xPy at p and $yR'x$ at p'.
   (ii) Individuals prefer x to y at p' if they prefer x to y or are indifferent between them at p.

If the dictator prefers y to x at p, the social preference must be yPx. Therefore, xP_iy or xI_iy for the dictator. Then, since he prefers x to y at p', this SWF can not be dictatorial.

(2) **Inversely dictatorial SWF case**: There is a case where for some pair of alternatives $x, y \in A$ we have two profiles p^1 and p^2 such that
  (i) The social preference is yP^1x at p^1 and xR^2y at p^2.
  (ii) Individuals prefer x to y at p^2 if they prefer x to y or are indifferent between them at p^1.

If the inverse-dictator prefers y to x at p^1, the social preference must be xP^1y. Therefore, xP_i^1y or xI_i^1y for the inverse-dictator. Then, since he prefers x to y at p^2, this SWF can not be inversely dictatorial. □

Appendix

In this appendix we show that IIA with inverse weak Pareto optimality implies the inverse form of monotonicity. By similar procedures we can show that IIA with weak Pareto optimality implies the normal form of monotonicity [2] . Let p be a profile such that

(1) individuals in G: xP_iy,
(2) individuals in G': xI_iy,
(3) individuals in $N - (G \cup G')$: yP_ix,

and yPx. Let p' be a profile such that

(1) individuals in G: $xP_i'yP_i'z$,
(2) individuals in G': $xI_i'yP_i'z$,
(3) individuals in $N - (G \cup G')$: $yP_i'zP_i'x$.

By inverse weak Pareto optimality $zP'y$, and by IIA $yP'x$. Then, transitivity implies $zP'x$. Consider a profile p'' such that

(1) individuals in G and G': $xP_i''zP_i''y$,
(2) individuals in $N - (G \cup G')$: $zP_i''x$ and $zP_i''y$.

By inverse weak Pareto optimality $yP''z$, and by IIA $zP''x$. Then, transitivity implies $yP''x$. By IIA it implies the inverse form of monotonicity.

Example We present an example which shows that monotonicity is weaker than IIA. Suppose that there are three alternatives x, y and z, and there are several individuals $1, 2, \ldots, n$. Let individual 1 be the dictator. If the dictator is indifferent between two alternatives, for example, x and y, the social preference is determined by the Borda rule. This social welfare function satisfies monotonicity because in the definition of monotonicity it is assumed that no individual is indifferent between x and y, but it does not satisfy IIA [3] .

[2] From Malawski and Zhou (1994) we know that IIA with non-imposition implies weak Pareto optimality or inverse weak Pareto optimality.
[3] About this example we refer to Denicolò (1998).

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