ECONOMICS BULLETIN

Intergenerational preferences and sensitivity to the present

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Abstract

This paper discusses the existence of binary relations on the set of intergenerational welfare paths in an infinite horizon setting. We show that any binary relation that satisfies anonymity and sup norm continuity is not sensitive to changes in welfare levels of the present generations. This result generalizes the impossibility theorems by Diamond [Econometrica 33 (1965) 170−177] and Sakai [Social Choice and Welfare, 16 (2003) 176−167].

Citation: Sakai, Toyotaka, (2003) "Intergenerational preferences and sensitivity to the present." *Economics Bulletin*, Vol. 4, No. 26 pp. 1−05
Submitted: May 18, 2003. **Accepted:** May 20, 2003.
URL: http://www.economicsbulletin.com/2003/volume4/EB−03D60001A.pdf

1 Introduction

This paper discusses the existence of binary relations on the set of intergenerational welfare (e.g., utility or consumption) paths in an infinite horizon setting. It is known that if a binary relation in an infinite horizon setting satisfies some anonymity axiom and continuity in some topology, then it violates many appealing axioms based on efficiency or equity concepts (Diamond, 1965; Campbell, 1985; Fleurbaey and Michel, 1997; Lauwers, 1997a; Shinotsuka, 1997; Sakai, 2003a,b). In particular, Diamond (1965) shows that there is no binary relation that satisfies anonymity, sup norm continuity, and strong monotonicity, and Sakai (2003a) shows that his result holds when strong monotonicity is replaced with a distributive fairness axiom, distributive fairness semiconvexity. We propose a new axiom called sensitivity to the present, which requires a binary relation to be sensitive to changes in welfare levels of the present generations. Strong monotonicity implies sensitivity to the present, and distributive fairness semiconvexity implies sensitivity to the present as well. We show that any binary relation that satisfies anonymity and sup norm continuity violates sensitivity to the present. Therefore, our result generalizes the impossibility theorems by Diamond and Sakai. Logical relations between sensitivity to the present and other known axioms are also clarified.

2 Notation

Our setting is based on Diamond (1965). Let N be the set of all positive integers and $X \equiv \{\mathsf{x} = (x_1, x_2, \ldots, x_t, \ldots) |$ for each $t \in \mathsf{N}, x_t \in [0,1]\}$. A path is an element $\mathsf{x} \equiv (x_1, x_2, \ldots, x_t, \ldots)$ of X, where each x_t is interpreted as the welfare level (e.g., the utility level or the amount of a single consumption good) of generation $t \in \mathsf{N}$. A binary relation on X is a subset $\succsim$ of $X \times X$. It is interpreted as a welfare criterion to evaluate paths in X. A finite permutation is a bijection π from N onto itself such that there is $t' \in \mathsf{N}$ satisfying $t = \pi(t)$ for each $t \geq t'$. Let Π be the set of all π. For each $\pi \in \Pi$ and each $\mathsf{x} \in X$, let $\pi(\mathsf{x}) \equiv (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(t)}, \ldots)$. Define the sup norm on X by $||\mathsf{x}||_\infty \equiv \sup x_t$ for $\mathsf{x} \in X$. For each $\mathsf{x}, \mathsf{y} \in X$ and each $t' \in \mathsf{N}$, define the compound path $(\mathsf{x}^{t'}, {}^{t'+1}\mathsf{y}) \in X$ by

$$(\mathsf{x}^{t'}, {}^{t'+1}\mathsf{y})_t \equiv x_t \text{ for each } t \leq t',$$

$$(\mathsf{x}^{t'}, {}^{t'+1}\mathsf{y})_t \equiv y_t \text{ for each } t \geq t'+1.$$

Compound paths that consist of more than two paths are defined in a similar way and denoted by, for example, $(\mathsf{x}^{t'}, {}^{t'+1}\mathsf{y}^{t''}, {}^{t''+1}\mathsf{z}^{t'''}, {}^{t'''+1}\mathsf{w})$.

Next, we state axioms on binary relations. The first one is anonymity, which requires equality in treating generations.

Anonymity: For each $\mathsf{x} \in X$ and each $\pi \in \Pi$, $\mathsf{x} \sim \pi(\mathsf{x})$.

As a continuity requirement, we impose sup norm continuity. We refer to Campbell (1985) and Lauwers (1997a) for discussions of continuity and to Lauwers (1997a) and Sakai (2003a) for discussions of the sup norm topology in this context. [1]

Sup norm continuity: For each $\mathsf{x}, \mathsf{y} \in X$ and each sequence $\{\mathsf{x}^n\}_{n=1}^{\infty}$ in X such that $\lim ||\mathsf{x}^n - \mathsf{x}||_\infty = 0$, if for each n, $\mathsf{y} \% \mathsf{x}^n$, then $\mathsf{y} \% \mathsf{x}$; if for each n, $\mathsf{x}^n \% \mathsf{y}$, then $\mathsf{x} \% \mathsf{y}$.

The next one is a condition of sensitivity to changes in welfare levels of the present generations. If a binary relation violates the condition, then there is a path such that any change of welfare levels of the present generations is ignored in welfare judgment at the path.

Sensitivity to the present: For each $\mathsf{x} \in X$, there are $\mathsf{y}, \mathsf{z} \in X$ and $t \in \mathsf{N}$ such that $(\mathsf{y}^t, {}^{t+1}\mathsf{x}) \succ (\mathsf{z}^t, {}^{t+1}\mathsf{x})$.

Our main theorem shows that any binary relation on X that satisfies anonymity and sup norm continuity violates sensitivity to the present. Since sensitivity to the present is a new axiom in this study, it is interesting to examine its logical relations to other axioms established so far.

The next axiom is straightforward:

Strong monotonicity: For each $\mathsf{x}, \mathsf{y} \in X$, if $\mathsf{x} \neq \mathsf{y}$ and for each $t \in \mathsf{N}$, $x_t \geq y_t$, then $\mathsf{x} \succ \mathsf{y}$.

The following fairness axiom (Sakai, 2003a) requires that at least one balanced path be preferable over two biased paths in the following manner: for two paths that are the same up to a finite permutation, there is at least one path that is a convex combination of the two paths and is preferable to the two.

Distributive fairness semiconvexity: For each $\mathsf{x} \in X$ and each $\pi \in \Pi$ with $\mathsf{x} \neq \pi(\mathsf{x})$, there is $s \in (0,1)$ such that $s\mathsf{x} + (1-s)\pi(\mathsf{x}) \succ \mathsf{x}$ and $s\mathsf{x} + (1-s)\pi(\mathsf{x}) \succ \pi(\mathsf{x})$.

The following non-dictatorship axioms (Chichilnisky, 1996) requires that evaluation of paths do not depend only on the welfare levels of present (future) generations.

[1] Diamond (1965), Chichilnisky (1996), Lauwers (1997a,b), Fleurbaey and Michel (1997), and Sakai (2003a) also use sup norm continuity. Continuity with respect to other topologies can be found in Diamond (1965), Svensson (1980), Brown and Lewis (1981), Campbell (1985), Lauwers (1997a), Fleurbaey and Michel (1997), and Shinotsuka (1998).

Non-dictatorship of the present: The following is not true: For each $\mathsf{x}, \mathsf{y} \in X$, if $\mathsf{x} \succ \mathsf{y}$, then there exists $t' \in \mathsf{N}$ such that for each $\mathsf{z}, \mathsf{w} \in X$ and each $t \geq t'$, $(\mathsf{x}^t, {}^{t+1}\mathsf{z}) \succ (\mathsf{y}^t, {}^{t+1}\mathsf{w})$.

Non-dictatorship of the future: The following is not true: For each $\mathsf{x}, \mathsf{y} \in X$, if $\mathsf{x} \succ \mathsf{y}$, then there exists $t' \in \mathsf{N}$ such that for each $\mathsf{z}, \mathsf{w} \in X$ and each $t \geq t'$, $(\mathsf{z}^t, {}^{t+1}\mathsf{x}) \succ (\mathsf{w}^t, {}^{t+1}\mathsf{y})$.

The Rawlsian criterion $\succsim^R$ on X is given by $\mathsf{x} \succsim^R \mathsf{y} \Leftrightarrow \inf x_t \geq \inf y_t$ for each $\mathsf{x}, \mathsf{y} \in X$.

Proposition 1. (1) Strong monotonicity implies sensitivity to the present,
(2) Distributive fairness semiconvexity implies sensitivity to the present,
(3) Sensitivity to the present implies non-dictatorship of the future,
(4) Anonymity and sensitivity to the present together imply non-dictatorship of the present,
(5) The Rawlsian criterion satisfies anonymity, sup norm continuity, non-dictatorship of the present, and non-dictatorship of the future, but not sensitivity to the present. [2]

Proof: (1) and (2) are trivial. Let us show (3). Let $\succsim$ be a binary relation on X that satisfies sensitivity to the present. There are $\mathsf{x}, \mathsf{y}, \mathsf{z} \in X$ and $t \in \mathsf{N}$ such that $(\mathsf{y}^t, {}^{t+1}\mathsf{x}) \succ (\mathsf{z}^t, {}^{t+1}\mathsf{x})$. If $\succsim$ violates non-dictatorship of the future, then since $(\mathsf{y}^t, {}^{t+1}\mathsf{x}) \succ (\mathsf{z}^t, {}^{t+1}\mathsf{x})$, there is a large $t' > t$ such that $\mathsf{x} = (\mathsf{x}^{t'}, {}^{t'+1}\mathsf{x}) \succ (\mathsf{x}^{t'}, {}^{t'+1}\mathsf{x}) = \mathsf{x}$, a contradiction.

Let us show (4). Suppose not, there is a binary relation $\succsim$ on X that satisfies anonymity and sensitivity to the present, but violates non-dictatorship of the present. By sensitivity to the present, there are $\mathsf{x}, \mathsf{y}, \mathsf{z} \in X$ and $t' \in \mathsf{N}$ such that $(\mathsf{y}^{t'}, {}^{t'+1}\mathsf{x}) \succ (\mathsf{z}^{t'}, {}^{t'+1}\mathsf{x})$. Since $\succsim$ violates non-dictatorship of the present, then for a large $t'' \in \mathsf{N}$,

$$(\mathsf{y}^{t'}, {}^{t'+1}\mathsf{x}^{t''}, {}^{t''+1}\mathsf{z}^{t''+t'}, {}^{t''+t'+1}\mathsf{x}) \succ (\mathsf{z}^{t'}, {}^{t'+1}\mathsf{x}^{t''}, {}^{t''+1}\mathsf{y}^{t''+t'}, {}^{t''+t'+1}\mathsf{x}).$$

This contradicts anonymity.

Finally, let us show (5). Clearly, the Rawlsian criterion $\succsim^R$ satisfies anonymity and sup norm continuity. Define $\mathbf{0} \equiv (0, 0, 0, \dots) \in X$. For each $\mathsf{x}, \mathsf{y} \in X$ with $\mathsf{x} \succ \mathsf{y}$ and each $t \in \mathsf{N}$, $(\mathbf{0}^t, {}^{t+1}\mathsf{x}) \sim^R (\mathbf{0}^t, {}^{t+1}\mathsf{y})$. So $\succsim^R$ satisfies non-dictatorship of the future. Also, for each $\mathsf{x}, \mathsf{y} \in X$ and each $t \in \mathsf{N}$, $(\mathsf{x}^t, {}^{t+1}\mathbf{0}) \sim^R (\mathsf{y}^t, {}^{t+1}\mathbf{0})$. So $\succsim^R$ violates sensitivity to the present. $\blacksquare$

[2] Chichilnisky (1996, Theorem 1) investigates whether or not many known welfare criteria, including the Rawlsian criterion, satisfy non-dictatorship of the present, non-dictatorship of the future, and strong monotonicity. Sakai (2003a, Proposition 2) characterizes a class of binary relations that satisfy transitivity, completeness, and non-dictatorship of the future. A more general statement of (4) can be found in Sakai (2003a, Proposition 1).

3 The impossibility theorem

Theorem 1. If a binary relation on X satisfies anonymity and sup norm continuity, then it violates sensitivity to the present.

Before proving this theorem, we shall give corollaries.

Corollary 1 (Diamond, 1965, Second impossibility theorem). [3] There is no binary relation on X that satisfies anonymity, sup norm continuity, and strong monotonicity.

Corollary 2 (Sakai, 2003a, Theorem 2). There is no binary relation on X that satisfies anonymity, sup norm continuity, and distributive fairness semiconvexity. [4]

The axioms in Theorem 1 are tight: A binary relation defined by discounting sum satisfies sup norm continuity and strong monotonicity; The overtaking criterion satisfies anonymity and strong monotonicity (Svensson, 1980, Theorem 3); Proposition 1 (5) implies that Theorem 1 does not hold if sensitivity to the present is relaxed to any one of the non-dictatorship axioms.

Let us prove Theorem 1. Hereafter, for notational simplicity, for each $n \in \mathbb{N}$, let $G(n)$ denote the finite sequence $\frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \dots, \frac{2^n}{2^n}$. Using $G(n)$'s, some paths are expressed. For example,

$$(x_1, G(1), G(2), x_8, x_9, \dots) = (x_1, \frac{1}{2}, \frac{2}{2}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, x_8, x_9, \dots).$$

The following lemma is the key to Theorem 1.

Lemma 1. Let $x, y \in [0,1]$ and $T \in \mathbb{N}$. Define

$$\mathsf{x} \equiv (x, G(T), G(T+1), G(T+2), \dots) \text{ and } \mathsf{y} \equiv (y, G(T), G(T+1), G(T+2) \dots).$$

Then, for each $\varepsilon > 0$, there is $\pi \in \Pi$ such that $||\pi(\mathsf{x}) - \mathsf{y}||_\infty < \varepsilon$.

Proof of Lemma 1: Let $x, y \in [0,1]$ and $T \in \mathbb{N}$. Without loss of generality, assume $x < y$. Take $\varepsilon > 0$ arbitrary. Let $n \in \mathbb{N}$ be such that $\frac{1}{2^n} < \min\{\varepsilon, \frac{1}{2}(y-x)\}$. Let $m(x)$ be the smallest integer that satisfies $x \leq \frac{m(x)}{2^n}$ and $m(y)$ the largest integer that satisfies $\frac{m(y)}{2^n} \leq y$. Since $\frac{1}{2^n} < \frac{1}{2}(y-x)$, then $\frac{m(x)}{2^n} < \frac{m(y)}{2^n}$. Hence $m(x) \leq m(y) - 1$. Define $\pi \in \Pi$ as follows:
$\pi(t) = 1 + \sum_{i=T}^{n-1} 2^i + m(x) \qquad \text{if } t = 1,$

[3] This theorem is due to M.E.Yaari (pp.176, Diamond 1965).

[4] This theorem was originally shown in the set of nonnegative bounded sequences, l^∞_+, however without any modification, his results also holds in X.

$= 1 \quad$ if $t = 1 + \sum_{i=T}^{n-1} 2^i + m(y)$,
$= t+1 \quad$ if $t = 1 + \sum_{i=T}^{n-1} 2^i + m(x), \dots, 1 + \sum_{i=T}^{n-1} 2^i + m(y) - 1$,
$= t \quad$ otherwise.

Thus,

$$\pi(z) = (\frac{m(y)}{2^n}, G(T), G(T+1), G(T+2) \dots, G(n-1),$$

$$\underbrace{\frac{1}{2^n}, \dots, \frac{m(x)-1}{2^n}, x, \frac{m(x)}{2^n}, \frac{m(x)+1}{2^n}, \dots, \frac{m(y)-1}{2^n}, \frac{m(y)+1}{2^n}, \dots, \frac{2^n}{2^n}}_{2^n},$$

$$G(n+1), G(n+2), \dots).$$

Since $||\pi(x) - y||_\infty \leq \frac{1}{2^n}$, then $||\pi(x) - y||_\infty < \varepsilon$. ¥

Proof of Theorem 1: Let $\succsim$ be a binary relation on X that satisfies anonymity and sup norm continuity. Define the path

$$x \equiv (G(1), G(2), G(2), G(3), G(3), G(3), \dots, \underbrace{G(n), G(n), \dots, G(n)}_{n \text{ number of } G(n)\text{'s}}, \dots).$$

Let us show that $\succsim$ violates sensitivity to the present. Let $y, z \in X$ and $t \in \mathbb{N}$. For each $T \geq t$, the path $(y^t, {}^{t+1}x)$ contains T number of $G(T)$'s. Also, for each $T \geq t$, the path $(z^t, {}^{t+1}x)$ contains T number of $G(T)$'s. So by Lemma 1, for each $n \in \mathbb{N}$, there is $\pi_n \in \Pi$ such that $||\pi_n((y^t, {}^{t+1}x)) - (z^t, {}^{t+1}x)||_\infty < \frac{1}{n}$. By anonymity, for each $n \in \mathbb{N}$, $\pi_n((y^t, {}^{t+1}x)) \sim (z^t, {}^{t+1}x)$. Since the sequence $\{\pi_n((y^t, {}^{t+1}x))\}_{n=1}^\infty$ converges to $(z^t, {}^{t+1}x)$ in the sup norm, then by sup norm continuity, $(y^t, {}^{t+1}x) \sim (z^t, {}^{t+1}x)$. So $\succsim$ violates sensitivity to the present. ¥

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