E C O N O M I C S   B U L L E T I N

On measuring changes in welfare when changes in consumption bundles are small

Indranil Dutta
University of East−anglia

Pattanaik Prasanta
University of California, Riverside

R. Robert Russell
University of California, Riverside

Abstract

The paper examines three extensively used propositions regarding changes in utility and social welfare attributable to small changes in consumption bundles. It is shown that, though these propositions are valid when the changes in consumption bundles are ``infinitesimally small,'' none of the propositions can be sustained when we have finite changes in consumption bundles, even when such finite changes are arbitrarily small. This drastically reduces the usefulness of the propositions for any practical purpose. The paper discusses the underlying formal structure of the problem and identifies the reason why the propositions run into difficulties in the case of finite changes.

Citation: Dutta, Indranil, Pattanaik Prasanta, and R. Robert Russell, (2003) "On measuring changes in welfare when changes in consumption bundles are small." *Economics Bulletin,* Vol. 4, No. 27 pp. 1−9
Submitted: June 4, 2003. **Accepted:** June 5, 2003.
URL: http://www.economicsbulletin.com/2003/volume4/EB−03D60002A.pdf

1. Introduction.

The purpose of this note is to reconsider some widely applied propositions about changes in utility and social welfare resulting from "small" changes in consumption bundles. Specifically, we investigate the "national income test" (see Varian, 1992, pp. 407-409) which, given an increase in real national income at the initial prices, is often used to infer an increase in potential welfare when the changes in consumption bundles are finite but "small." Starting with a discussion of the national income test for a one-consumer economy, we then proceed to examine, for the general case of many consumers, two different versions of the test—one that assumes the initial distribution of goods to be socially optimal and another that does not make this assumption. We seek to demonstrate that none of several propositions involving the national income test is tenable for finite changes in consumption bundles, no matter how small these changes may be, and that the propositions are, therefore, of doubtful value for any practical purpose. We also discuss the underlying formal structure of the problem and identify the flaw in the reasoning that is sometimes advanced in support of these results in the context of finite changes.

The first proposition that we examine in this paper relates to the change in a competitive consumer's utility when there is a small change in his/her consumption bundle. As Varian (1992, p.409) puts it, the claim is that "small changes in [a consumer's consumption bundle] are preferred or not preferred [by the consumer] as the change in the value of the bundle is positive or negative."[1] The following is a somewhat more explicit statement of the proposition.

> Proposition A: Let x_i be the optimal consumption bundle of a competitive consumer, i, in an initial situation where the price vector is p. Let x_i' be another consumption bundle such that $p \cdot x_i' > p \cdot x_i$. Then, if x_i' is "sufficiently close" to x_i, the consumer strictly prefers x_i' to x_i.

If valid, in certain situations Proposition A would allow one to infer, on the basis of a very limited amount of information, an increase in the welfare of a competitive consumer. Given only that, at the initial prices, the value of the new commodity bundle consumed by the competitive consumer is higher than the value of the initial consumption bundle, Proposition A would let us conclude that there is an increase in the consumers welfare, provided the change in the consumers consumption bundle is "small." However, as we will show later, this conclusion cannot be sustained for finite changes in the consumption bundle, however small.

Proposition A, in its turn, constitutes the basis of another proposition which provides a sufficient condition for "potentially Pareto preferred"[2] social changes and is widely used to assess the effect of "small changes" in policy. In Varian's (1992, p. 408) words, Proposition B tells us that, "if ... national income at the original prices increases ...then it must be possible

[1] In what follows, we often refer to the discussion of these claims in Varian (1992). This is partly for the sake of convenience, Varian (1992) being one of the most authoritative graduate textbooks on microeconomic theory, and partly because Varian (1992) gives one of the clearest statements of these propositions.

[2] Cf. Varian (1992, p. 408).

to increase every agent's utility," given that the change under consideration is sufficiently small. A somewhat more explicit statement of this proposition may be given as follows.

> Proposition B: Suppose, we have n competitive consumers who, given the price vector p, are initially optimising subject to their relevant budget constraints. Now consider a new situation such that the value, at p, of the new aggregate consumption bundle is higher than the value of the initial aggregate consumption bundle. If the new consumption bundle of every consumer is "sufficiently close" to his/her initial consumption bundle, then it is possible to redistribute the new aggregate consumption bundle so as to make every consumer better off as compared to the initial situation.

The next proposition involves the assumption of a socially optimal distribution of income in the initial situation and provides a sufficient condition for an actual increase in social welfare. It may be stated as follows.

> Proposition C: Assume that the initial situation is a competitive equilibrium where the income distribution is socially optimal. Starting from this initial situation, if there is an increase in national income at the initial prices, then, provided the new consumption bundle of every consumer is "sufficiently close" to his/her original consumption bundle, social welfare in the new situation must be higher than social welfare in the initial situation.

Note that each of the three propositions involves the stipulation that the new consumption bundle(s) must be sufficiently close to the original consumption bundle(s). How do we interpret the term "sufficiently close"? One possible interpretation is that the change from the initial consumption bundle to the new consumption bundle is infinitesimally small, so that the propositions are really about the signs of the relevant derivatives/differentials at the initial situation rather than about the effects of finite changes. Under this interpretation, all three propositions are valid (we comment further on this in Section 5). However, any practical application of these propositions has to be necessarily in a context where the changes in consumption bundles are finite. It would, therefore, be tempting to assume that the propositions are also valid when the term "sufficiently close" is interpreted as referring to arbitrarily small finite changes. Unfortunately, as we show, none of the propositions remains valid for finite changes in consumption bundles, no matter how small these changes are assumed to be.

The plan of the paper is as follows. In Section 2, we lay down the few notations that we need. In Section 3, we give counterexamples to show that none of the three propositions is valid for finite changes in consumption bundles, even when these changes are stipulated to be arbitrarily small. In Section 4, we analyse the general structure of the problem and identify the lacuna in the reasoning that is sometimes given for these claims in the context of finite changes. We conclude in Section 5.

2. Notation.

Let R be the set of real numbers and R_+ be the set of non-negative real numbers. Let n be the number of consumers in the economy and m be the number of commodities ($m > 1$).

For every consumer i, the consumption set is assumed to be R^m_+. For all $z, z' \in R^m_+$, $h(z, z')$ denotes the (Euclidean) distance between z and z'. The consumption bundles of consumer i are denoted by x_i, x'_i, z_i, z'_i, etc. The prices are all assumed to be positive. The price vectors are denoted by p, p', etc. Every consumer i is assumed to have a direct utility function $u^i(x_i)$ satisfying the properties of continuity, strict quasi-concavity and strict monotonicity. Consumer i's wealth is denoted by the non-negative real numbers, w_i, w'_i, etc.

3. Some counterexamples.

We first construct counterexamples to show that none of the three propositions mentioned in Section 1 is valid for finite, though arbitrarily small, changes in consumption bundles. The following proposition will be useful in this context.

Proposition 1: Let p be the price vector prevailing in the initial situation, where x_i $(x_i \neq 0)$ is an optimal consumption bundle of a competitive consumer, i. Then, for every positive real number g, there exists $\overset{*}{x}_i \in R^m_+$ such that

$$[h(x_i, \overset{*}{x}_i) < g;\ p \cdot \overset{*}{x}_i > p \cdot x_i;\ \text{and } u^i(\overset{*}{x}_i) < u^i(x_i)]. \tag{1}$$

Proof: Let g be any given positive number. Let

$$H(p, x_i) := \{z_i \in R^m_+ \mid p \cdot z_i = p \cdot x_i\},$$

and let

$$B(x_i) := \{z_i \in R^m_+ \mid u^i(z_i) \geq u^i(x_i)\}.$$

Given strict quasi-concavity of the utility function, there cannot be more than one optimal consumption bundle for the consumer. Therefore, x_i must be the unique optimal consumption bundle in the initial situation, and we must have $H(p, x_i) \cap B(x_i) = \{x_i\}$. This implies that

$$\forall\ z_i \in H(p, x_i),\ z_i \neq x_i \implies u^i(z_i) < u^i(x_i). \tag{2}$$

Note that, since $x_i \neq 0$, there exists $x'_i \in H(p, x_i)$ such that $x_i \neq x'_i$. Consider such x'_i. Since $[x_i \in H(p, x_i),\ x'_i \in H(p, x_i)$, and $x_i \neq x'_i]$, the open interval, $]x_i, x'_i[$, is non-empty, and, for all $x''_i \in]x_i, x'_i[$, $x''_i \in H(p, x_i)$. Hence, there must exist $\overset{*}{z}_i \in H(p, x_i)$ such that

$$[0 < h(x_i, \overset{*}{z}_i) < g/2]$$

Consider such $\overset{*}{z}_i \in H(p, x'_i)$. Since $\overset{*}{z}_i \in H(p, x_i)$, $p \cdot \overset{*}{z}_i = p \cdot x_i$. Also, since $x_i \neq \overset{*}{z}_i \in H(p, x_i)$, by (2), we have $u^i(\overset{*}{z}_i) < u^i(x_i)$.

Now consider $z'_i \in R^m_+$, such that $p \cdot z'_i > p \cdot x_i$. Since the prices are positive, such z'_i exists. Since $p \cdot z'_i > p \cdot x_i$ and $p \cdot \overset{*}{z}_i = p \cdot x_i$,

$$z''_i \in]z'_i, \overset{*}{z}_i[\implies p \cdot z''_i > p \cdot x_i. \tag{3}$$

Since $u^i(\overset{*}{z}_i) < u^i(x)$, by the continuity of the utility function, there exists $\overset{*}{x}_i \in]z'_i, \overset{*}{z}_i[$, such that $h(\overset{*}{z}_i, \overset{*}{x}_i) < g/2$ and $u^i(\overset{*}{x}_i) < u^i(x_i)$. Since $h(x_i, \overset{*}{z}_i) < g/2$ and $h(\overset{*}{z}_i, \overset{*}{x}_i) <$

$g/2$, $h(x_i, \overset{*}{x}_i) < g$. Also, given $\overset{*}{x}_i \in]z_i' \ , \overset{*}{z}_i[$, by (3), $p \cdot \overset{*}{x}_i > p \cdot x_i$. Thus, (1) holds for $\overset{*}{x}_i$. This completes the proof.

The basic point of Proposition 1 can be illustrated in a simple diagram for the case of two commodities. In the Figure (located at the end of the paper), the initial optimum consumption bundle of consumer i is given by x_i, where the initial budget frontier, AB, is tangent to an indifference curve. Now consider the circle, with x_i for its centre, which is shown in the figure. It is clear that there are points in the interior of the circle that are above the budget frontier and, hence, cost more than x_i at the initial prices, and are below the indifference curve, IC, passing through x_i. These are the points in the shaded area of the circle. What happens if we make the circle arbitrarily small? Since the indifference curve IC is strictly convex to the origin and since x_i is the point of tangency between AB and IC, it is clear that, irrespective of how small the circle is, we can always find points in the interior of the circle, which are above AB and below IC. Proposition 1 tells us that, essentially, the same result holds in the general case.

In view of Proposition 1, Proposition A is clearly non-tenable for finite changes in the consumer's consumption bundle, even when we assume that these finite changes are arbitrarily small.[3]

Usually, Proposition A constitutes the basis of Proposition B (see, for example, Varian (1992, p.409)), and it is easy to show that Proposition B runs into the same problem as Proposition A. To see this, consider first the (trivial) case of a one-consumer economy. Let p be the initial price vector and let x_1 be the initial optimal consumption bundle of the single consumer in the economy. Then, by Proposition 1, for every positive number g, we can find $\overset{*}{x}_1$ such that (1) holds. Obviously, in this one-consumer economy, there is no redistribution of the aggregate consumption bundle $\overset{*}{x}_1$ that will make the single consumer better off as compared to x_1. While, formally, this is a valid counterexample for Proposition B, it involves the restrictive feature that there is only one consumer in the economy. To construct a counterexample with n consumers ($n > 1$), consider a competitive, pure exchange economy with n consumers such that, for each consumer i, the utility function is given by $u^i(z_i) = z_{i1}^{1/2} + z_{i2}^{1/2} + \cdots + z_{im}^{1/2}$. Suppose the initial endowment bundle of each consumer is given by w, where w contains one unit of each commodity. Then, it is easy to see that $\big(p, (\overbrace{w, \ldots, w}^{n \text{ times}})\big)$, where $p_j = 1$ for every commodity j , is a competitive equilibrium for this economy. The aggregate consumption bundle in this equilibrium is, of course, nw. Let g be any positive number. Then, noting that all consumers have identical preferences and identical consumption bundles, w, in the initial situation, by Proposition 1, we can find a commodity bundle w' such that w' contains a positive amount of each commodity; $h(w, w') < g$; $p \cdot w' > p \cdot w$; and for every consumer i, $u^i(w') < u^i(w)$. Now, consider a new situation where every consumer consumes w', so that the new aggregate consumption

[3] Of course, given strong monotonicity, the following is true: Let p be the price vector prevailing in the initial situation where the optimum consumption bundle of competitive consumer i is given by x_i ($x_i \neq 0$). Then, for every positive real number g, there exists $\overset{*}{x}_i \in R_+^m$ such that $[h(x_i, \overset{*}{x}_i) < g$; $p \cdot \overset{*}{x}_i > p \cdot x_i$; and $u^i(\overset{*}{x}_i) > u^i(x_i)]$. However, this is clearly very different from Proposition A.

bundle is nw'. By construction, $p\cdot(nw') > p\cdot(nw)$. However, the new aggregate consumption bundle, nw', cannot be redistributed among the consumers to make everybody better off as compared to the initial situation. This can be shown as follows. When every consumer consumes w', the marginal rate of substitution between every pair of commodities is the same for all consumers (recall that all consumers have identical preferences). Hence, the allocation where everybody gets w' is a Pareto optimal distribution of the aggregate bundle nw', and no redistribution of nw' can make everyone better off as compared to $(\overbrace{w', \ldots, w'}^{n \text{ times}})$. Since it is impossible to redistribute nw' to make every consumer better off as compared to $(\overbrace{w', \ldots, w'}^{n \text{ times}})$, it must be impossible to distribute nw' to make everybody better off as compared to the allocation $(\overbrace{w, \ldots, w}^{n \text{ times}})$, which is Pareto superior to $(\overbrace{w', \ldots, w'}^{n \text{ times}})$.

To see the difficulty with Proposition C in the case of finite changes, consider a competitive equilibrium where the price vector is p, the consumption bundles are $x_1, \ldots, x_n$, and the distribution of income is socially optimal. Let g be any given positive number. Then, by Proposition 1, one can find consumption bundles $\overset{*}{x}_1, \ldots, \overset{*}{x}_n$ such that, for every consumer i, (1) holds. Clearly, the situation with consumption bundles $\overset{*}{x}_1, \ldots, \overset{*}{x}_n$ is socially inferior to the initial situation under every Paretian social welfare function.

4. The general structure of the problem.

It should not come as a surprise that, when the changes in consumption bundles are finite, one can construct counterexamples for Proposition A, and hence for the other two propositions based on Proposition A. To see this, it may be helpful to consider the general, formal structure of our problem. Given that x_i is consumer i's optimal consumption bundle in the initial situation where the price vector is p, $\alpha_i \; p \cdot (x_i' - x_i) = \nabla u^i(x_i) \cdot (x_i' - x_i)$, where α_i is i's marginal utility of wealth. Since α_i is positive (by the assumption of strict monotonicity of u^i), $p \cdot (x_i' - x_i)$ and $\nabla u^i(x_i) \cdot (x_i' - x_i)$ have the same sign, and the problem of inferring the sign of $[u^i(x_i') - u^i(x_i)]$ from the sign of $[p \cdot (x_i' - x_i)]$ is, therefore, the same as the problem of inferring the sign of $[u^i(x_i') - u^i(x_i)]$ from the sign of the linear approximation, $[\nabla u^i(x_i) \cdot (x_i' - x_i)]$, of $[u^i(x_i') - u^i(x_i)]$. The question, therefore, is whether one can infer anything about the sign of $[u^i(x_i') - u^i(x_i)]$ given the sign of its linear approximation $[\nabla u^i(x_i) \cdot (x_i' - x_i)]$.

In general, given an arbitrarily specified differentiable function $F: R_+^m \to R$, the sign of $[\nabla F(y) \cdot (y' - y)]$, the linear approximation of $[F(y') - F(y)]$, does not tell us anything about the sign of $[F(y') - F(y)]$, no matter how close y' is to y. However, curvature assumptions on F can yield definite implications. The following result follows in a straightforward fashion from the well-known differential characterizations of curvature of level sets of F.

<u>Proposition 2</u>: If F is strictly quasi-concave, $\nabla F(y) \neq 0$, and $y \neq y'$, then

$$[\nabla F(y) \cdot (y' - y) \leq 0] \implies [F(y') < F(y)] \tag{4}$$

and, hence,

$$[F(y') \geq F(y)] \Longrightarrow [\nabla F(y) \cdot (y' - y) > 0]. \tag{5}$$

Weakening the curvature condition from strict quasi-concavity to quasi-concavity in Proposition 2 converts the strict inequality in (4) to a weak inequality ((5), which is the contrapositive of (4), then changes appropriately). However, it is not possible to strengthen either (4) or (5) by strengthening strict quasi-concavity to strict concavity in Proposition 2.

It is also impossible to establish the reverse of the implication, (4) and (5) in Proposition 2. However, the reasoning advanced in the literature in support of Proposition A in the context of finite changes in consumption bundles essentially amounts to a claim that, when a function F on R^m_+ is strictly quasi-concave and $y \neq y'$, we can have the reverse of the implication in (5), provided y and y' are sufficiently close. For example, Varian's (1992, p. 408-409)) justification for Proposition A in the context of finite changes in consumption bundles runs as follows. Given the strictly quasi-concave utility function u^i, Varian points out that, when the switch from the initial consumption bundle x_i to x'_i is small, using a first-order Taylor series approximation, we have

$$u^i(x'_i) - u^i(x_i) \approx \nabla u^i(x_i) \cdot (x'_i - x_i) = \alpha_i \; p \cdot (x'_i - x_i) \tag{6}$$

(recall that α_i is i's (positive) marginal utility of wealth in the initial situation). Given (6), it is asserted that, if $p \cdot (x'_i - x_i)$ and, hence, the linear approximation, $[\nabla u^i(x_i).(x'_i - x_i)]$, of $[u^i(x'_i) - u^i(x_i)]$, is positive, then so is $[u^i(x'_i) - u^i(x_i)]$. The problem with this reasoning is that, while, by taking x'_i sufficiently close to x_i, one can make the value of $\big[[u^i(x'_i) - u^i(x_i)] - [\nabla u^i(x_i).(x'_i - x_i)]\big]$ arbitrarily small, it is still possible to have $[u^i(x'_i) - u^i(x_i)]$ negative even though $[\nabla u^i(x_i) \cdot (x'_i - x_i)]$ happens to be positive. In fact, as Proposition 1 tells us, in every neighbourhood, however small, of x_i, we can find x'_i such that this "perverse" possibility holds. This is just an example of our earlier general observation, namely, that it is not possible to reverse implication (5) in Proposition 2, no matter how close y' may be to y. Is it possible to formulate a linear approximation condition which will be sufficient for an increase in the utility of a consumer? Proposition 2 tells us that, for a strictly quasi-concave function, a linear approximation condition is sufficient for a decrease in the value of the function and necessary for an increase in its value, while, for strictly quasi-convex functions, a linear approximation condition is sufficient for an increase in the value of the function and necessary for a decrease.

5. Derivatives and finite changes.

We noted earlier that differential variants of Propositions A, B, and C are valid. Indeed, it seems to us that it is these valid results about the signs of relevant derivatives and differentials, which have given rise to invalid beliefs about the effects of finite changes with which economists are inevitably confronted in applied areas such as cost-benefit analysis. In the context of cost-benefit analysis, let b denote the size of a project, and, for every consumer

i and every commodity k, let x_i be a function, $x_i(b)$, of b, where, given the size b, $x_i(b)$ is determined through the general competitive equilibrium. Then

$$du^i/db = \sum_k u^i_k[dx_{ik}/db] = \sum_k \alpha_i p_k[dx_{ik}/db] = \alpha_i \sum_k p_k[dx_{ik}/db], \tag{7}$$

where, as before, α_i is i's marginal utility of wealth. Thus, $du^i/db > 0$ if and only if $\sum_k p_k[dx_{ik}/db] > 0$. While this is certainly true, it does not justify the claim that, starting from an initial size $\overset{*}{b}$, if we increase the project size by a finite amount, and if, at the initial prices, the value of the new consumption bundle of consumer i is higher than the value of $x_i(\overset{*}{b})$ bundle, then i must be better off after the change, provided the (finite) change in the consumption bundle is sufficiently small: our Proposition 1 shows that this is not necessarily true, no matter how small the change in the consumption bundle may be. Similarly, letting $dx(b)$ denote $\sum_k dx_{ik}(b)$, it can be shown that, if $p \cdot dx(b) > 0$, then $dx(b)$ can be distributed to make every consumer better off than in the initial situation.[4] But this is of little help in practical cost-benefit analysis, where, inevitably, we have to consider finite changes in consumption bundles: however small these finite changes may be, there is always the possibility that the value, at initial prices, of the incremental aggregate consumption bundle is positive and, yet, the change is not potentially Pareto superior to the original situation.

The basic point here is quite general,[5] though it is sometimes overlooked when intuitive interpretations are given for comparative static results derived in terms of derivatives or differentials. Consider a function $f(x)$ where x and $f(x)$ take real (scalar) values. Suppose it is known that $f'(\overset{*}{x})$ is positive, and this is all that is known about the function. While the information that $f'(\overset{*}{x})$ is positive may be interesting, by itself it does not tell us anything about what happens to $f(x)$ when, starting from $\overset{*}{x}$, there is a "very small" finite increase in the value of x. This is because it is possible that, for every positive number t, however small, one can find a real (scalar) valued function of x such that the function has a positive first derivative at $\overset{*}{x}$ but has a lower value at some point in the interval $[\overset{*}{x}, \overset{*}{x} + t]$ than at $\overset{*}{x}$. Of course, if $f'(x)$ is positive everywhere (as in the standard comparative-static results in consumer and producer theory), then we can be sure that an increase in the value of x will lead to an increase in the value of $f(x)$.

References.

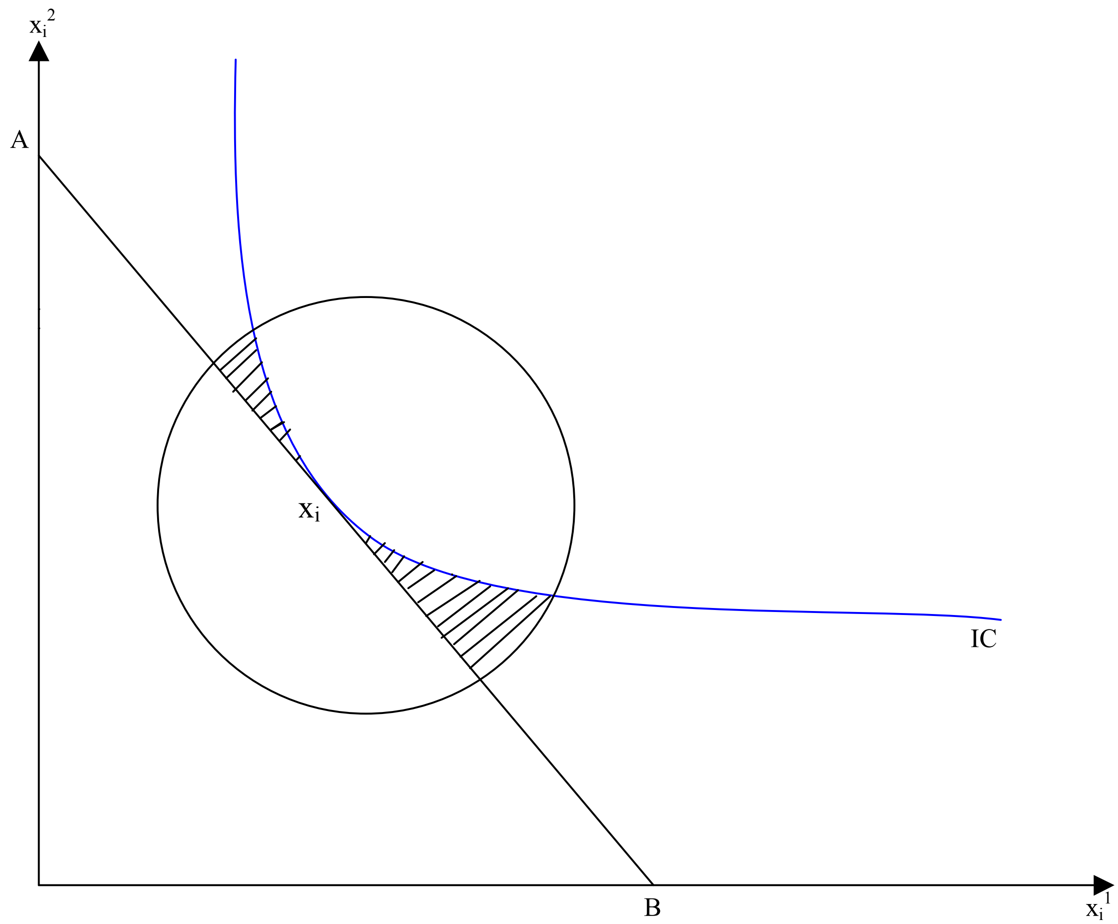
Bruce, N., and R. G. Harris (1982) "Cost-Benefit Criteria and the Compensation Principle in Evaluating Small Projects" *Journal of Political Economy* **90**, 755–776.

Samuelson, P. A. (1947) *Foundations of Economic Analysis*, Harvard University Press: Cambridge, Massachussetts.

Varian, H. R. (1992) *Microeconomic Analysis*, Third Edition, W.W. Norton and Company: New York.

[4] See Theorem 1 of Bruce and Harris (1982).

[5] See Samuelson (1947, pp. 46–48).



FIGURE