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Indeterminacy of equilibrium price of money, market price of risk and interest rates

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Abstract

This paper shows that a market price of nominal risk plays an important role in the determinacy of the price of money under a stochastic continuous−time monetary economy. It is presented that a sufficient condition for the determinacy of the price of money is either an exogenously given nominal short rate or an exogenously given market price of nominal risk, which implies that a different market price of nominal risk may yield a different price of money. Thus the nominal pricing kernel is not endogenously determined under the original assumptions of the model. If central banks can determine not only the money supply but also the nominal short rate process, the policy leads to the determinacy of a process of the price of money and the pricing kernels. Explicit solutions for real quantities and nominal quantities are provided for two cases of utility functional forms.

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1. Introduction

The purpose of this paper is (*i*) to show that in general it is not possible to derive the price of money endogenously in a stochastic monetary economy with money-in-the-utilities, and (*ii*) to present a way to overcome the issue of the indeterminacy of the price of money, real interest rates and nominal interest rates from financial economics point of view. We follow the stochastic continuous-time setting of Basak and Gallmeyer (1999) [1] and extend their results on the price of money to more general cases in the closed monetary economy.

With paying more attention to the important role of the market prices of the risks we study the endogenous relationship among real quantities and nominal quantities. Then we show that a sufficient condition for the determinacy of the price of money is either (but not both) an exogenously given nominal short rate or an exogenously given market price of nominal risk in addition to the nominal money supply and the consumption commodity endowment. The equation for the price of money should be solved *backwards* from the transversality condition. Our idea is that if the diffusion term is given, then the price of money can be obtained accordingly, which implies that a different market price of nominal risk may yield a different price of money. It leads to the indeterminacy in the model. There are two implications from our results that are not observed in the deterministic cases. First, the nominal pricing kernel is not endogenously determined under the original assumptions of the model. Secondly, if central banks can determine not only the money supply but also the nominal short rate process, the policy leads to the determinacy of the price of money and the pricing kernels. Explicit solutions for real quantities and nominal quantities are provided for two cases of utility functional forms.

2. The Model

2.1 Basic Setting and Pricing Kernels

In this subsection we setup the model by following Basak and Gallmeyer (1999) and then briefly review the basic results of the real pricing kernel and the nominal pricing kernel.[2] We consider a closed monetary economy where a representative agent is endowed with a perishable consumption commodity δ and the nominal money supply M and maximizes his lifetime utility under certain budget constraints in an infinite time horizon. The uncertainities in the economy are caused by an n-dimensional Brownian motion W on a complete filtered probability space $(\Omega,\ \mathcal{F},\ P,\ \{\mathcal{F}_t\})$ where P is some objective probability measure. The filtration $\{\mathcal{F}_t\}$ is the augmented filtration generated by W.

[1] Bakshi and Chen (1996) solves the price of money with a logarithm utility function. Basak and Gallmeyer (1999) considers a two-country model with more general utility functions. It is easily checked that their discussion can be applied to our model.

[2] The terms state-price density or stochastic discount factor are often used synonymously with pricing kernel. The real pricing kernel is used for the cash flow in real term while the nominal pricing kernel is used for the cash flow in nominal term.

The endowment δ in the commodity is given by an Ito process

$$d\delta(t) = \delta(t)\big[\mu_\delta(t)dt + \sigma_\delta(t)dW(t)\big], \quad \delta(0) = \delta_0,$$

where μ_δ and σ_δ are respectively $\mathbb{R}$-valued and $\mathbb{R}^n$-valued processes. (Hereafter we assume that these coefficient processes of dt and $dW(t)$ have appropriate dimensions and conditions.) The nominal money supply M is also exogenously given by

$$dM(t) = M(t)\big[\mu_M(t)dt + \sigma_M(t)dW(t)\big], \quad M(0) = M_0.$$

The price of money q is supposed to follow $dq(t) = q(t)\big[\mu_q(t)dt + \sigma_q(t)dW(t)\big]$, which will be determined later.

We assume that the representative agent can trade n securities without any friction: (i) a real money account which pays the real short rate r continuously as dividend in the unit of the commodity, (ii) a nominal money account which pays the nominal short rate R continuously as dividend in the unit of money, (iii) a stock which pays the commodity δ and (iv) zero-coupon bonds with distinct $n-3$ maturities. Net supplies of these securities in this economy are assumed to be zero. R is assumed to follow an Ito process $dR(t) = R(t)[\mu_R(t)dt + \sigma_R(t)dW(t)]$. The price of the stock at time t is denoted by $S(t)$.

Given the price of money q in units of the commodity, the representative agent chooses the consumption c and the nominal money balance M^D to maximize a life utility

$$E\Big[\int_0^\infty e^{-\rho t}u(c(t), q(t)M^D(t))dt\Big]$$

subject to the budget constraint

$$E\Big[\int_0^\infty z(t)(c(t) + R(t)q(t)M^D(t))dt\Big] \leq S(0) + E\Big[\int_0^\infty z(t)R(t)q(t)M(t)dt\Big],$$

where z is the real pricing kernel. $u(c, m)$ is assumed to be increasing, strictly concave and three times continuously differentiable in both arguments. Notice that we assume and impose the transversality condition

$$\lim_{T\to\infty} E\Big[z(T)q(T)M^D(T) \mid \mathcal{F}_t\Big] = 0 \tag{1}$$

in the budget constraint of the maximization problem, and we can show that the static budget constraint above is equivalent to the dynamic constraint that the consumption at each instant is financed by the endowment, the money supply and a portfolio of securities as argued in Basak and Gallmeyer (1999) Proposition 2.1.

By the first order conditions of the maximization problem and the general equilibrium conditions $c = \delta$ and $M^D = M$, it is well-known that the real pricing kernel z is proportional to the marginal utility of consumption and the nominal short rate is the marginal rate of substituion between the commodity and the real money balance,

$$\begin{aligned} z(t) &= e^{-\rho t}\frac{u_c(\delta(t), q(t)M(t))}{u_c(\delta(0), q(0)M(0))}, & (2)\\ R(t) &= \frac{u_m(\delta(t), q(t)M(t))}{u_c(\delta(t), q(t)M(t))}. & (3) \end{aligned}$$

We can define the nominal pricing kernel Z as

$$Z(t) = z(t)q(t)/q(0). \tag{4}$$

We call λ and Λ respectively the market price of real risk and the market price of nominal risk. Then the pricing kernels are characterised by the pair of the short rate and the market price of the risk,

$$\begin{aligned} dz(t) &= -z(t)\big[r(t)dt + \lambda(t)dW(t)\big], \quad z(0) = 1, & (5)\\ dZ(t) &= -Z(t)\big[R(t)dt + \Lambda(t)dW(t)\big], \quad Z(0) = 1. & (6) \end{aligned}$$

If the process of the real money $m = qM$, which is not yet determined, is assumed to follow $dm(t) = m(t)\big[\mu_m(t)dt + \sigma_m(t)dW(t)\big]$, we have the following general expressions by applying Ito's formula to equations (2) and (4) and comparing with (5) and (6),

$$\begin{aligned} r(t) &= \rho + A_{cc}(t)\mu_\delta(t)\delta(t) + A_{cm}(t)\mu_m(t)m(t) - \frac{1}{2}\Big(A_{ccc}(t)\|\sigma_\delta(t)\delta(t)\|^2 \\ &\quad + 2A_{ccm}(t)\sigma_\delta(t)\sigma_m(t)\delta(t)m(t) + A_{cmm}(t)\|\sigma_m(t)m(t)\|^2\Big), & (7)\\ R(t) &= r(t) - \mu_q(t) + \sigma_q(t)\lambda(t), & (8)\\ \lambda(t) &= A_{cc}(t)\sigma_\delta(t) + A_{cm}(t)\sigma_m(t), & (9)\\ \Lambda(t) &= \lambda(t) - \sigma_q(t), & (10) \end{aligned}$$

where $\quad A_{ij}(t) = -\dfrac{u_{ij}(\delta(t), m(t))}{u_c(\delta(t), m(t))}, \quad A_{ijk}(t) = \dfrac{u_{ijk}(\delta(t), m(t))}{u_c(\delta(t), m(t))},$

and $\|\cdot\|$ represents an n-dimensional Euclidean norm.

Notice that equation (7) represents the real interest rate, as being well-known, as a sum of (*i*) the time-preference, (*ii*) the risk aversion coefficient times the expected growth rate and (*iii*) the effect of precautionary savings. Equation (8) is the modified Fisher equation since the expected inflation rate is $-\mu_q(t)$ and the covariance term appears on the right-hand side.

2.2 Indeterminacy due to a volatility

Our concern is whether the price of money q can be expressed with exogenous variables δ and M. By expressing equation (6) in the integral form and taking the expectation we see that

$$q(t) = \frac{1}{z(t)}E\Big[\int_t^\infty z(s)q(s)R(s)ds \mid \mathcal{F}_t\Big] + \frac{1}{z(t)}\lim_{T\to\infty} E\Big[z(T)q(T) \mid \mathcal{F}_t\Big], \tag{11}$$

which states that the money is priced as a security paying the dividend stream $q(s)R(s)$ as the liquidity service. Equation (11) should be solved backwards from the terminal value since the process q appears on the both sides. The first term of the right-hand side of (11) can be recognised as the forward-looking (or fundamental) term and the second term as the bubble term.

Regarding to the bubble term, if $p(t,T)$ denotes the time t price of the nominal zero coupon bond with the maturity date T, the bubble term is proportional to $\lim_{T\to\infty} p(t,T)$ which we assume is zero,

$$\frac{1}{z(t)} \lim_{T\to\infty} E\Big[z(T)q(T) \mid \mathcal{F}_t\Big] = \frac{q(t)}{Z(t)} \lim_{T\to\infty} E\Big[Z(T) \mid \mathcal{F}_t\Big] = q(t) \lim_{T\to\infty} p(t,T) = 0. \quad (12)$$

The assumption is economically desirable and reasonable in the sense that (i) deflationary paths are ruled out and (ii) the nominal short rates are always positive from the assumptions on the utility function and equation (3).[3]

It is important to note that the forward-looking term may depend on the volatilities of the price of money.[4] Our basic idea is that *if the volatility of zq that is the market price of nominal risk is given, then the equation* (11) *should be solvable for zq.* In other words a different volatility process will give a different value of the forward-looking term. This indeterminacy is not observed in the deterministic cases. Intuitively the indeterminacy of the price of money is similar to the fact that in Black-Scholes model the volatilities should be exogenously given to calculate an option premium.

Thus we will look for a solution by assuming the market price of nominal risk is exogenously given somehow. In addition to the standing assumptions (1) (for the optimal real money balance) and (12) (for the positive nominal interest rates), for the simplicity of our discussion we focus on the cases with the further transversality condition of

$$\lim_{T\to\infty} E\Big[\big(z(T)q(T)F(T)\big)^k \mid \mathcal{F}_t\Big] = 0 \text{ for all } t \in [0,\infty) \quad (13)$$

$$\text{where} \qquad F(t) = \exp\Big(\frac{1}{2}\int_0^t \|\Lambda(u)\|^2 du + \int_0^t \Lambda(u) dW(u)\Big)$$

for some positive constant k.[5] With this condition we can solve (11) for utility functions with some functional forms as shown in the subsequent section. The following auxiliary lemma is useful to get the solution. The essence of the proof of Lemma 1 is that the process XG does not have a diffusion term so that the

[3] The assumption makes the nominal pricing kernel a potential. A potential X is a right-continuous nonnegative supermartingale which satisfies $\lim_{t\to\infty} E(X_t) = 0$. See Jin and Glasserman (2001).

[4] An exception is a case of a logarithm utility function $u(c,m) = \beta \ln c + (1-\beta)\ln m$ in which case q does not appear in the forward-looking term. See Bakshi and Chen (1996).

[5] Condition (13) is purely for technical convenience to obtain a simple closed-form solution. The relationship with (12) is not clear due to the term of $F(T)$. In principle it is possible to solve with other forms of the transversality condition though the results will be different accordingly.

equation becomes an ordinary differential equation with respect to time t which can be easily solved.

Lemma 1. *Let $X = (X(t); 0 \le t \le \infty)$ be a stochastic process satisfying a stochastic differential equation* [6]

$$dX(t) = -f(t)X(t)^{1-k}dt + X(t)\sigma(t)dW(t),$$

with $\lim_{T\to\infty} E\Big[\big(X(T)G(T)\big)^k \mid \mathcal{F}_t\Big] = 0$ *for all* $t \in [0,\infty)$, *where f and σ are known integrable stochastic processes, $k > 0$ is a constant, and*

$$G(t) = \exp\Big(\frac{1}{2}\int_0^t \|\sigma(u)\|^2 du - \int_0^t \sigma(u)dW(u)\Big).$$

Then the solution is given by

$$X(t) = \Big(kE\Big[\int_t^\infty f(s)\Big(\frac{G(s)}{G(t)}\Big)^k ds \mid \mathcal{F}_t\Big]\Big)^{1/k}.$$

Proof. See Appendix A. □

2.3 Equilibrium in the case of a separable utility function

In this subsection we seek solutions by assuming the separable utility function

$$u(c,m) = \beta\frac{c^{1-\gamma}-1}{1-\gamma} + (1-\beta)\frac{m^{1-\alpha}-1}{1-\alpha}, \tag{14}$$

where α, β and γ are constants with $\alpha > 0, \beta \in (0,1), \gamma > 0$.

In the case of the separable utility, from equations (2), (7) and (9), real quantities are determined independently of the price of money as

$$\begin{aligned} z(t) &= e^{-\rho t}\delta(t)^{-\gamma}/\delta(0)^{-\gamma}, \quad \lambda(t) = \gamma\sigma_\delta(t), \\ r(t) &= \rho + \gamma\mu_\delta(t) - \frac{\gamma(1+\gamma)}{2}\|\sigma_\delta(t)\|^2. \end{aligned}$$

Equation (3) is reduced to $R(t) = \dfrac{1-\beta}{\beta}\dfrac{\delta(t)^\gamma}{q(t)^\alpha M(t)^\alpha}$ therefore equation (6) implies

$$d(zq)(t) = -\frac{1-\beta}{\beta}\frac{\delta(0)^\gamma e^{-\rho t}}{z(t)^{1-\alpha}M(t)^\alpha}\big(z(t)q(t)\big)^{1-\alpha}dt - z(t)q(t)\Lambda(t)dW(t). \tag{15}$$

[6] A solution of the equation means a process X satisfying

$$X(t) = \int_t^\infty f(s)X(s)^{1-k}ds - \int_t^\infty X(s)\sigma(s)dW(s), \quad \forall t \in [0,\infty).$$

This solution is a special case of a solution of a backward stochastic differential equation in the literature such as Yong and Zhou (1999) because the diffusion term σ is already known in this problem.

Since z is known as above, the price of money q can be obtained if a solution zq of equation (15) exists. Lemma 1 ensures that this is the case if the market price of nominal risk Λ is specified. Furthermore if the nominal short rate R, instead of Λ, is specified, then q is also determined as in Proposition 1. Two equations (16) and (17) below link three variables, q, R and Λ.

Proposition 1. *If the market price of nominal risk Λ is exogenously given, then the price of money and the nominal short rate satisfying* (13) *with $k = \alpha$ is determined by*

$$q(t) = \Big[\frac{1-\beta}{\beta}\frac{\delta(t)^{\gamma}}{R(t)}\Big]^{1/\alpha}\frac{1}{M(t)}, \tag{16}$$

$$\frac{1}{R(t)} = \alpha E\Big[\int_t^{\infty} K(t,s)\exp\Big(-\int_t^s b_1(u)du\Big)ds \mid \mathcal{F}_t\Big], \tag{17}$$

where

$$\begin{aligned} K(t,s) &= \exp\Big[-\frac{1}{2}\int_t^s \|\sigma_R(u)\|^2 du + \int_t^s \sigma_R(u)dW(u)\Big], \\ \sigma_R(u) &= (1-\alpha)\lambda(u) + \alpha\Lambda(u) - \alpha\sigma_M(u), \\ b_1(u) &= \rho - (1-\alpha)r(u) + \alpha\mu_M(u) - \frac{1}{2}\big((1-\alpha)\|\lambda(u)\|^2 + \alpha\|\Lambda(u)\|^2 \\ &\quad + \alpha\|\sigma_M(u)\|^2 + \|(1-\alpha)\lambda(u) + \alpha\Lambda(u) - \alpha\sigma_M(u)\|^2\big), \end{aligned}$$

if the solutions satisfy (1) *and* (12).

Conversely, if the nominal short rate process R, that is strictly positive, is exogenously given, then the price of money q is determined by equation (16) *and the market price of nominal risk Λ must satisfy equation* (17).

Proof. See Appendix B. □

There are a couple of implications. Notice that the first two observations below are not seen in the deterministic economies that do not involve the market prices of risks.

First, Proposition 1 states that the nominal pricing kernel is not endogenously determined without a further specification, and once either variable of three variables q, R and Λ is specified then the remaining two variables can be determined. The market price of nominal risk plays a critical role in the determinacy of the price of money that bridges the gap between real quantities and nominal ones.

Secondly, some policy rules can be embeded into the model in order to obtain the price of money or the market price of nominal risk. The rules may include Taylor rule or a direct control of the nominal short rate. If central banks can determine not only the money supply but also the nominal short rate process, the policy leads to the determinacy of a process of the price of money (an inflation process) and the pricing kernels.

Thirdly, the result is a generalization of Basak and Gallmeyer (1999) and Bakshi and Chen (1996). If one wants to see deterministic R ($\sigma_R(t) \equiv 0$) and $b_1(t)$

is assumed to be a positive constant $b > 0$, then $R(t) = b\alpha^{-1}$ that is the corresponding result to one obtained in Basak and Gallmeyer (1999) Proposition 4.2. If coefficients in the processes of δ and M are assumed to be constant and one sets $\alpha = \gamma = 1$ and $\Lambda(t) = \sigma_M$, then $q(t) = \frac{1-\beta}{\beta}\frac{\delta(t)}{(\rho + \mu_M - \sigma_M^2)M(t)}$ as is shown in Bakshi and Chen (1996) Theorem 3.

Remark 1. Lemma 1 is so useful that, instead of the separable utility (14), the similar discussion can be applied to the following non-separable utility function

$$u(c,m) = \frac{1}{1-\gamma}\Big(\Big[c^{\beta}m^{1-\beta}\Big]^{1-\gamma} - 1\Big),$$

where β and γ are constants with $0 < \beta < 1, 0 < \gamma < (2-\beta)/(1-\beta)$.

Since this utility function has non-zero cross derivatives, real quantities are affected by nominal quantities and the real and nominal pricing kernels should be solved simultaneously as opposed to the separable utility. Even though this problem looks more complicated, it is possible to solve equations simultaneously for both real quantities and nominal quantities in the similar way as Proposition 1 by using Lemma 1. The corresponding formula are obtained as follows with $\kappa = (1-\beta)(1-\gamma)$.

$$\begin{aligned} q(t) &= \frac{1-\beta}{\beta}\frac{\delta(t)}{R(t)M(t)}, \\ \frac{1}{R(t)} &= \frac{1}{1+\kappa}E\Big[\int_t^{\infty} K(t,s)\exp\Big(-\int_t^s b_2(u)du\Big)ds \mid \mathcal{F}_t\Big], \end{aligned}$$

where

$$\begin{aligned} K(t,s) &= \exp\Big[-\frac{1}{2}\int_t^s \|\sigma_R(u)\|^2 du + \int_t^s \sigma_R(u)dW(u)\Big], \\ \sigma_R(u) &= \frac{1}{1+\kappa}\Big[(1-\gamma)\sigma_\delta(u) + \Lambda(u) - \sigma_M(u)\Big], \\ b_2(u) &= \frac{1}{1+\kappa}\Big[\rho - (1-\gamma)\big(\mu_\delta(u) + \frac{1}{2}\|\sigma_\delta(u)\|^2\big) - \frac{1}{2}\|\Lambda(u)\|^2 \\ &\quad + \mu_M(u) + \frac{1}{2}\|\sigma_M(u)\|^2 - \frac{1}{2(1+\kappa)}\|(1-\gamma)\sigma_\delta(u) + \Lambda(u) - \sigma_M(u)\|^2\Big]. \end{aligned}$$

3. Conclusion

We demontrate the very close relationship between the price of money, the nominal short rate and the market price of nominal risk. When a solution is built from the transversality condition backwards, the volatility term is required. However the market price of nominal risk is not endogenously determined with endowment processes. It suggests that the market price of nominal risk is an equivalently important factor as the other two variables, the price of money and the nominal short rate, in the context of monetary and fiscal policy. The determinacy of the yield curve of nominal interest rates depends on the specification of these policies, which is left for future research.

References

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Appendix

A Proof of Lemma 1

Since G satisfies $dG(t) = G(t)\|\sigma(t)\|^2 dt - G(t)\sigma(t)dW(t)$, it can be easily checked that the diffusion term of the process XF disappears and the evolution of XF can be written as an ordinary differential equation

$$\frac{d(XG)(t)}{dt} = -f(t)G(t)^k(XG)(t)^{1-k}.$$

The solution is

$$X(T)^k G(T)^k - X(t)^k G(t)^k = -k\int_t^T f(s)G(s)^k ds.$$

Taking the conditional expectation $E[\cdot|\mathcal{F}_t]$ and making $T \to \infty$ yields the result.

B Proof of Proposition 1

By setting

$$X(t) = z(t)q(t), \quad f(t) = \frac{1-\beta}{\beta}\frac{\delta(0)^\gamma e^{-\rho t}}{z(t)^{1-\alpha}M(t)^\alpha}, \quad k = \alpha,$$

and applying Lemma 1 with equation (14) we have

$$z(t)q(t) = \Big[\alpha E\Big[\int_t^\infty \frac{1-\beta}{\beta}\frac{\delta(0)^\gamma e^{-\rho s}}{z(s)^{1-\alpha}M(s)^\alpha}\Big(\frac{F(s)}{F(t)}\Big)^\alpha ds \mid \mathcal{F}_t\Big]\Big]^{1/\alpha},$$

where $F(t) = \exp\Big(\frac{1}{2}\int_0^t \|\Lambda(u)\|^2 du + \int_0^t \Lambda(u)dW(u)\Big)$. Thus

$$q(t) = \Big[\frac{1-\beta}{\beta}\frac{\delta(0)^\gamma e^{-\rho t}}{z(t)M(t)^\alpha}\alpha E\Big[\int_t^\infty e^{-\rho(s-t)}\Big(\frac{z(t)}{z(s)}\Big)^{1-\alpha}\Big(\frac{M(t)}{M(s)}\Big)^\alpha\Big(\frac{F(s)}{F(t)}\Big)^\alpha ds \mid \mathcal{F}_t\Big]\Big]^{1/\alpha}.$$

Recalling the solution $Y = \{Y(t)\}$ of the stochastic differential equation $dY(t) = Y(t)[\mu_Y(t)dt + \sigma_Y(t)dW(t)]$ satisfies

$$Y(s) = Y(t)\exp\Big(\int_t^s \big(\mu_Y(u) - \frac{1}{2}\|\sigma_Y(u)\|^2\big)du + \int_t^s \sigma_Y(u)dW(u)\Big),$$

the terms in the integrand can be written as

$$e^{-\rho(s-t)}\frac{F(s)^\alpha}{z(s)^{1-\alpha}M(s)^\alpha} = \frac{F(t)^\alpha}{z(t)^{1-\alpha}M(t)^\alpha}K(t,s)\exp\Big(-\int_t^s b_1(u)du\Big).$$

For the latter part of the proposition, when R and z are known, the price of money is given by equation (16) from equation (3). Then

$$Z(t) = z(t)q(t)/q(0) = e^{-\rho t}\left[\frac{\delta(0)}{\delta(t)}\right]^{\gamma(1-1/\alpha)}\left[\frac{R(0)}{R(t)}\right]^{1/\alpha}\frac{M(0)}{M(t)}$$

By comparing the diffusion terms on both sides, we have

$$\Lambda(t) = \frac{1}{\alpha}\Big(-(1-\alpha)\gamma\sigma_\delta(t) + \sigma_R(t) + \alpha\sigma_M(t)\Big).$$

Applying Lemma 1 for $X(t) = Z(t)$ with equation (6) gives the equation (17).