ECONOMICS BULLETIN

Existence and monotonicity of optimal debt contracts in costly state verification models

Ludovic Renou
European University Institute and CODE, Universitat Autonoma de Barcelona

Guillaume Carlier
Bordeaux I, MAB, UMR CNRS 5466 and Bordeaux IV, GRAPE, UMR CNRS 5113

Abstract

This note gives a simple proof of the existence and monotonicity of optimal debt contracts in simple models of borrowing and lending with ex−post asymmetric information, risk−averse agents and heterogeneous beliefs. Our argument is based on the concept of nondecreasing rearrangement and on a supermodular version of Hardy−Littlewood inequality.

Citation: Renou, Ludovic and Guillaume Carlier, (2003) "Existence and monotonicity of optimal debt contracts in costly state verification models." *Economics Bulletin,* Vol. 7, No. 5 pp. 1−9
Submitted: August 7, 2003. **Accepted:** September 11, 2003.
URL: http://www.economicsbulletin.com/2003/volume7/EB−03G00004A.pdf

1. Introduction

Consider a contracting problem between a borrower and a lender subject to a *costly state verification* (henceforth, CSV (see Townsend(1979)) problem, that is to say, the project return is costlessly observed by the borrower while the lender has to pay a monitoring cost. Assuming *risk neutrality* and *homogenous beliefs*, Gale and Hellwig (1985) and Williamson (1986) have shown that optimal debt contracts among the class of incentive compatible and individually rational contracts are *simple debt contracts* where the borrower repays a fixed interest rate whenever possible and the lender seizes all the profit realized when the borrower defaults. Assuming risk neutrality *but* heterogeneous beliefs, Carlier and Renou (2003) have shown that simple debt contracts do not necessarily survive as optimal contracts. These authors provide a sufficient condition for simple debt contracts to be optimal and show by means of counter-examples that this condition is easily violated. However, no proof of existence of optimal contracts was provided.

In this note, we go one step further. We prove the existence of optimal debt contracts without assuming risk neutrality and/or homogeneous beliefs. It is important to notice that our existence result also holds for the special cases studied in Gale and Hellwig (1985), Williamson (1986) or Carlier and Renou (2003). Moreover, we show that optimal contracts are monotone in that the higher the project return, the higher the repayment from the borrower to the lender is. On the technical side, our argument essentially relies on the concept of nondecreasing rearrangement and on a supermodular version of Hardy-Littlewood inequality; concepts that have been recently introduced in economics by Carlier and Dana (2002, 2003). More precisely, the main mathematical result we use is as follows: the mathematical expectation of a function U of two Borel functions x and y increases if we substitute x and y by their *nondecreasing rearrangements* $\tilde{x}$ and $\tilde{y}$ and if U satisfied a *supermodularity* condition.(see Theorem 1 for a precise formulation) We refer the interested reader to Carlier and Dana (2002, 2003) for more applications of these concepts.

Section 2 presents a simple CSV model and reviews results from the rearrangement techniques' literature. Section 3 proves the existence and monotonicity of optimal contracts. Finally, section 4 offers some concluding remarks.

2. A simple costly state verification model

2.1. The model

We consider a simple economy with a unique borrower and a unique lender. The lender is endowed with a single unit of a good that might be used for both investment and consumption while the borrower has no initial wealth. The borrower and the lender are expected utility maximizer with Bernoulli utility functions u and v, and we assume the following:

(A1) The utility function u and v: $\mathbb{R} \to \mathbb{R}_+$ are concave, increasing, twice continuously differentiable.

Moreover, the borrower has access to an investment project requiring exactly one unit of the good to be undertaken. The project return is a realization ω of the random variable $\widetilde{\omega}$ with full support on $[0, \overline{\omega}]$. Furthermore, the project return ω is costlessly observable only to the borrower while the lender has to pay a cost of γ to perfectly monitor the return.

Finally, the borrower and the lender might differ in their beliefs about the project return. The borrower believes that $\widetilde{\omega}$ admits the probability density function μ while the lender believes in ν. Density functions are common knowledge and continuous. In addition, we assume that the borrower believes the project profitable, that is $\int_0^{\overline{\omega}} \omega\mu(\omega)d\omega > u(0)$, and that one of the following holds

(A2) (resp, (A2')) The function $LR(\cdot) : \omega \mapsto LR(\omega) := \frac{\mu(\omega)}{\nu(\omega)}$ is differentiable and for all $(\omega, v) \in [0, \overline{\omega}]^2$ the following condition is satisfied:

$$-u''(\omega - v)LR(\omega) - u'(\omega - v)LR'(\omega) \geq 0. \text{ (resp, } > 0.)$$

Condition **(A2)** (resp, **(A2')**) simply states that the Arrow-Pratt coefficient of absolute risk aversion for the borrower is greater (resp, strictly greater) than the differential version of the likelihood ratio. In particular, **(A2)** holds in the special case of risk neutrality and homogenous beliefs or in the case of a monotone likelihood ratio (i.e., $LR'(\omega) \leq 0, \forall\omega$.)

Since the borrower has access to a high return project but does not have the wealth to undertake the project, he can propose a debt contract to the lender, as defined below.

Definition 1 *A debt contract is a pair* $\left(\widehat{R}, M\right)$ *where*

- $\widehat{R}:[0,\overline{\omega}]\rightarrow\mathbb{R}_+,\ \omega\mapsto\widehat{R}(\omega)$ *is the repayment in state* ω,
- $\widehat{R}(\omega)\leq\omega,\ \forall\omega\in[0,\overline{\omega}]$, *the limited liability constraint,*
- $M\subset[0,\overline{\omega}]$, *a subset of* $[0,\overline{\omega}]$ *where monitoring takes place.*

Definition 2 *A debt contract* $\left(\widehat{R},M\right)$ *is truthtelling if and only if* $\forall\omega\in[0,\overline{\omega}]$, $\forall\omega'\notin M$,

$$\widehat{R}(\omega)\leq\widehat{R}(\omega').$$

A debt contract is thus said to be truthtelling when the borrower has no incentive to misreport the project return. Without loss of generality, we restrict ourselves to the class of debt contracts which satisfy the direct revelation mechanism (see Townsend (1988)) and are truthtelling. Note that if $(\widehat{R},M)$ is truthtelling, then $\widehat{R}$ has to be constant on $[0,\overline{\omega}]\setminus M$, say with value $\overline{R}$, and obviously:[1]

$$\{\widehat{R}(\cdot)<\overline{R}\}\subseteq M\subset\{\widehat{R}(\cdot)\leq\overline{R}\}.$$

Changing $(\widehat{R},M)$ into the new truthtelling contract $(\widehat{R},\{\widehat{R}(\cdot)<\overline{R}\})$ is therefore weakly Pareto improving since this transformation does not increase the expected monitoring cost. Hence, a truthtelling contract is simply determined by a pair $(R(\cdot),\overline{R})$ where the threshold $\overline{R}$ determines monitoring states $M=\{R(\cdot)<\overline{R}\}$ and $0\leq R(\omega)\leq\omega$ for all $\omega\in[0,\overline{\omega}]$.

2.2. The borrower's program

Given a contract $(R(\cdot),\overline{R})$, the borrower's expected payoff is

$$\int_{\{R(\cdot)<\overline{R}\}}u(\omega-R(\omega))\mu(\omega)d\omega+\int_{\{R(\cdot)\geq\overline{R}\}}u(\omega-\overline{R})\mu(\omega)d\omega=\int_0^{\overline{\omega}}u(\omega-\min(R(\omega),\overline{R}))\mu(\omega)d\omega,$$

while the lender's expected payoff is

$$\int_{\{R(\cdot)<\overline{R}\}}v(R(\omega)-\gamma)\nu(\omega)d\omega+\int_{\{R(.)\geq\overline{R}\}}v(\overline{R})\nu(\omega)d\omega=\int_0^{\overline{\omega}}v(\min(R(\omega),\overline{R}))-\gamma\mathbf{1}_{\{R(.)<\overline{R}\}})\nu(\omega)d\omega.$$

[1] Hereafter, we denote $\{\widehat{R}(\cdot)<\overline{R}\}$ for the set $\{\omega\in[0,\overline{\omega}]:\widehat{R}(\omega)<\overline{R}\}$, etc.

Since the lenders reservation utility is $v(1)$, a contract $(R(\cdot), \overline{R})$ is accepted if the following participation constraint holds:

$$V(R(\cdot), \overline{R}) := \int_0^{\overline{\omega}} v(\min(R(\omega), \overline{R}) - \gamma \mathbf{1}_{\{R(\cdot)<\overline{R}\}})\nu(\omega)d\omega \geq v(1). \qquad (1)$$

Definition 3 *A contract $(R(\cdot), \overline{R})$ is admissible if it satisfies the participation constraint (1) and*

$$0 \leq R(\omega) \leq \omega, \text{ for all } \omega \in [0, \overline{\omega}].$$

It follows that the borrower's program is given by

$$\sup_{(R(\cdot),\overline{R})} U(R(\cdot), \overline{R}) := \int_0^{\overline{\omega}} u(\omega - \min(R(\omega), \overline{R}))\mu(\omega)d\omega, \qquad (2)$$

subject to $(R(\cdot), \overline{R})$ being *admissible*.

(A3) The set of admissible contracts $(R(\cdot), \overline{R})$ is nonempty.

2.3. Rearrangement inequalities and supermodularity

This section briefly reviews the *rearrangement techniques* introduced by Carlier and Dana (2002, 2003) in economics. We refer the interested reader to their work and the references cited therein for the proofs and further results (see also Carlier (2003) for more advanced results).

Let m be a probability measure on $[0, \overline{\omega}]$. Assume that m is absolutely continuous with respect to the Lebesgue measure and that it admits a continuous and positive density μ on $[0, \overline{\omega}]$.

Definition 4 *Two Borel functions on $[0, \overline{\omega}]$, x and y are equimeasurable with respect to m if and only if they fulfill one of the following equivalent conditions:*

1. *$m(x^{-1}(B)) = m(y^{-1}(B))$ for every Borel subset B of $\mathbb{R}$,*

2. *for every bounded Borel function f:*

$$\int_0^{\overline{\omega}} f(x(\omega))dm(\omega) = \int_0^{\overline{\omega}} f(y(\omega))dm(\omega).$$

Proposition 1 *Let m be as previously and R be any real-valued Borel function on $[0, \overline{\omega}]$. Then there exists a unique right-continuous nondecreasing function, $\widetilde{R}$, which is equimeasurable to R with respect to m.*

$\widetilde{R}$ is called the *nondecreasing rearrangement* of R with respect to the probability measure m. A key result of the rearrangement techniques is the celebrated Hardy-Littlewood inequality. Such inequalities state that integral of the form

$$\int_0^{\overline{\omega}} L(x(\omega), y(\omega))dm(\omega)$$

increases when we substitute the arbitrary functions $x(\cdot)$ and $y(\cdot)$ by their nondecreasing rearrangements $\widetilde{x}(\cdot)$ and $\widetilde{y}(\cdot)$ and L satisfied a *supermodularity* condition, as defined below.

Definition 5 *Let L be a function : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. L is called supermodular (respectively strictly supermodular) if for all $(x, h) \in \mathbb{R} \times \mathbb{R}_+$:*

$$y \mapsto [L(x + h, y) - L(x, y)]$$

is nondecreasing (respectively increasing) on $\mathbb{R}$.

It is worth noting that if L is $C^2(\mathbb{R}\times\mathbb{R}, \mathbb{R})$, then a necessary and sufficient condition for L to be supermodular is that the cross-derivative $\partial^2_{xy}L$ is nonnegative. In other words, L satisfies the so-called Spence-Mirlees single crossing condition. A few examples of supermodular functions are $L(x, y) = xy$, $L(x, y) = U(x - y)$ with U concave or $L(x, y) = g(x + y)$ with g convex. (see Topkis (1998, p43) for further examples of supermodular functions)

Theorem 1 *Let L be a continuous and supermodular function. Let x and y be two real-valued Borel functions defined on $[0, \overline{\omega}]$ and $\widetilde{x}(\cdot)$ and $\widetilde{y}(\cdot)$ be their nondecreasing rearrangement with respect to m. We have*

$$\int_0^{\overline{\omega}} L(\widetilde{x}(\omega), \widetilde{y}(\omega))dm(\omega) \geq \int_0^{\overline{\omega}} L(x(\omega), y(\omega))dm(\omega),$$

provided that the right-hand side of this inequality is well-defined. Furthermore if L is strictly modular and x is increasing then the previous inequality is an equality if and only if $y = \widetilde{y}$.

Finally, since most optimization problems in economics are constrained, we should pay attention to the stability of the set of constraints (typically, equality and inequality constraints) under nondecreasing rearrangements.

Lemma 1 *Let x be any real-valued Borel function defined on $[0, \overline{\omega}]$ and let $\widetilde{x}$ be its nondecreasing rearrangement with respect to m. Let f and g be two nondecreasing functions: $[0, \overline{\omega}] \to \mathbb{R}$. If $f \leq x \leq g$ $m - a.e.$, then $f \leq \widetilde{x} \leq g$ $m - a.e.$.*

3. Existence and monotonicity

Lemma 2 *Assume **(A1),(A2)**. Let $(R(\cdot),\overline{R})$ be a admissible contract and let $\widetilde{R}$ be the nondecreasing rearrangement of R with respect to $\nu(\omega)d\omega$. Then $(\widetilde{R}(\cdot),\overline{R})$ is admissible and $U(\widetilde{R}(\cdot),\overline{R})\geq U(R(\cdot),\overline{R})$. If, in addition, **(A2')** holds, this inequality is strict unless* $\min(R(\cdot),\overline{R})=\min(\widetilde{R}(\cdot),\overline{R})$.

Proof. Let $(R(\cdot),\overline{R})$ be a admissible contract and let $\widetilde{R}$ be the nondecreasing rearrangement of R with respect to $\nu(\omega)d\omega$. First observe that since $0\leq R(\omega)\leq\omega$, we have $0\leq\widetilde{R}(\omega)\leq\omega$ by Lemma 1. By equimeasurability of R and $\widetilde{R}$ with respect to $\nu(\omega)d\omega$, $V(\widetilde{R}(\cdot),\overline{R})=V(R(\cdot),\overline{R})$ hence $(\widetilde{R}(\cdot),\overline{R})$ is admissible.
Finally, by assumption **(A2)**, the function $(\omega,v)\mapsto u(\omega-v)LR(\omega)$ is supermodular. Hence by Theorem 1 one has

$$U(\widetilde{R}(\cdot),\overline{R})=\int_0^{\overline{\omega}}u(\omega-\min(\widetilde{R}(\omega),\overline{R}))LR(\omega)\nu(\omega)d\omega$$

$$\geq\int_0^{\overline{\omega}}u(\omega-\min(R(\omega),\overline{R}))LR(\omega)\nu(\omega)d\omega=U(R(\cdot),\overline{R}).$$

If, in addition, **(A2')** holds, then the function $(\omega,v)\mapsto u(\omega-v)LR(\omega)$ is strictly supermodular. By Theorem 1, it follows that the previous inequality is strict unless $\min(R(\cdot),\overline{R})$ is nondecreasing i.e. $\min(R(\cdot),\overline{R})=\min(\widetilde{R}(\cdot),\overline{R})$.

□

Thus, from Lemma 2, monotonicity is a necessary optimality condition for the borrower's program. We can therefore assume without loss of generality that the repayment function is nondecreasing i.e., $R(\cdot)$ is nondecreasing.

Theorem 2 *Assume **(A1),(A2), (A3)**. The borrower's program (2) has a solution $(R^*(\cdot),\overline{R}^*)$ with $R^*(\cdot)$ nondecreasing on $[0,\overline{\omega}]$. If, in addition, **(A2')** holds, then any solution $(R^*(\cdot),\overline{R}^*)$ to the borrower's program is nondecreasing on $\{R^*(\cdot)<\overline{R}^*\}$.*

Proof. Let $(R_n(\cdot),\overline{R}_n)_n$ be a maximizing sequence of (2). From Lemma 2, we can replace $R_n(\cdot)$ by its nondecreasing rearrangement with respect to $\nu(\omega)d\omega$. From the uniform boundedness of $(R_n(\cdot))_n$ and Helly's selection Theorem (see Appendix) some subsequence of $(R_n(\cdot))_n$ (not relabeled) converges pointwise to some nondecreasing function $R^*(\cdot)$. Similarly we may assume that $(\overline{R}_n)_n$ converges to some $\overline{R}^*$. Obviously $(R^*(\cdot), \overline{R}^*)$ satisfies the limited liability constraint and Lebesgue's dominated convergence Theorem implies that $U(R^*(\cdot), \overline{R}^*)$ is the value of problem (2). It remains to prove that $(R^*(\cdot), \overline{R}^*)$ satisfies the participation constraint. First note that:

$$\mathbf{1}_{\{R^*(\cdot)<\overline{R}^*\}} \leq \liminf \mathbf{1}_{\{R_n(\cdot)<\overline{R}_n\}}, \ \nu\text{-a.e.}$$

so that

$$v(\min(R^*(\cdot), \overline{R}^*) - \gamma \mathbf{1}_{\{R^*(\cdot)<\overline{R}^*\}}) \geq \limsup v(\min(R_n(\cdot), \overline{R}_n) - \gamma \mathbf{1}_{\{R_n(\cdot)<\overline{R}_n\}}).$$

Applying Fatou's Lemma, it follows that $(R^*(\cdot), \overline{R}^*)$ solves (2).
If, in addition, **(A2')** holds, then Lemma 2 implies that any solution $(R^*(\cdot), \overline{R}^*)$ of the borrower's program satisfies $\min(R(\cdot), \overline{R}) = \min(\widetilde{R}(\cdot), \overline{R})$ hence is non-decreasing on $\{R^*(\cdot) < \overline{R}^*\}$.

□

4. Concluding remarks

To conclude, we give a few remarks and hint further research. As has been already noticed, if we assume risk-neutrality ($u = v = Id$ with $Id : [0, \overline{\omega}] \to [0, \overline{\omega}]$, $\omega \mapsto \omega$) and homogeneous beliefs ($\mu = \nu$), then optimal debt contracts are *simple debt contracts* where $(R^*(\cdot), \overline{R}^*) = (Id, \overline{R}^*)$. (see Gale and Hellwig (1985) or Williamson (1986)) However, assuming risk-neutrality but heterogeneous beliefs, Carlier and Renou (2003) have shown that simple debt contracts do not necessarily survive as optimal contracts. These authors provide a sufficient condition for simple debt contracts to be optimal and show by means of counter-examples that this condition is easily violated. Our existence and monotonicity result is certainly a first step towards a complete characterization of optimal debt contracts in a CSV model with risk-aversion and heterogeneity of beliefs, but further research is still needed. It is believed that obtaining further analytical results would be extremely difficult, if not out of reach, and that numerical analysis might give more insights. Finally, it is hoped that rearrangement techniques would prove to be fruitful for solving other optimization problems.

5. Appendix: Helly selection Theorem

We recall a compactness property of nondecreasing functions due to Helly. (see for instance Natanson (1955))

Theorem 3 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a uniformly bounded sequence of nondecreasing functions on* $[0,\overline{\omega}]$*. Then there exists a nondecreasing function* x *defined on* $[0,\overline{\omega}]$ *and a subsequence, again denoted* $(x_n)_{n\in\mathbb{N}}$*, which converges pointwise to* x *on* $[0,\overline{\omega}]$*.*

References

Carlier, G. (2003) "Extremal properties of comonotone pairs of random variable and applications" Mimeo, Université de Bordeaux.

Carlier, G., and R.A. Dana. (2002) "Rearrangement inequalities in non convex economic models" *Cahier du Ceremade 0238*.

Carlier, G., and R.A. Dana. (2003) "Existence and monotonicity of solutions to moral-hazard problems" *Cahier du Ceremade 0323*.

Carlier, G., and L, Renou. (2003) "A costly state verification model with diversity of opinions" *Mimeo*, Université de Bordeaux and European University institute.

Gale D., and M. Hellwig. (1985) "Incentive-Compatible Debt Contracts: The One-Period Problem" *Review of Economics Studies*, **52(4)**, 647-663.

Natanson, I.P. (1955) "Theory of functions of a real-variable," Frederick Ungar Publishing Co., New York.

Topkis, D.M. (1998) "Supermodularity and complementarity" Princeton University Press.

Townsend, R. (1979) "Optimal Contracts and Competitive Markets with Costly State Verification" *Journal of Economic Theory*, **21**, 265-93.

Townsend, R. (1988) "Information Constrained Insurance: The Revelation Principle Extended" *Journal of Monetary Economics*, **21**, 411-450.

Williamson, S.D. (1986) "Costly Monitoring, Financial Intermediation and Equilibrium Credit Rationing" *Journal of Monetary Economics*, **18(2)**, 159-179.