E C O N O M I C S B U L L E T I N

Malthus and Solow − a note on closed−form solutions

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Abstract

Recently, Jones (2002} and Barro and Sala−í−Martin (2004) pointed out that the neoclassical growth model with a Cobb−Douglas technology has a closed−form solution. This note makes a similar remark for the Malthusian model: I develop and characterize a closed−form solution. Moreover, I emphasize structural similarities between the Malthusian and the neoclassical model if the dynamic behavior is governed by a Bernoulli differential equation.

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1 Introduction

It has recently been pointed out that the neoclassical growth model of Solow (1956) and Swan (1956) has a closed-form solution if the aggregate production function is Cobb-Douglas. Jones (2002) and Barro and Sala-í-Martin (2004) added this result to the second edition of their books on economic growth.

This note makes a similar remark for the Malthusian model that depicts the interplay between population growth and per-capita income. I show that an appropriately chosen Malthusian population equation in conjunction with a Cobb-Douglas production technology allows for a closed-form solution. Moreover, I highlight several striking similarities to the neoclassical growth model. First, I emphasize the similar role of diminishing returns in both models. They affect the death rate of the population in the Malthusian setting and the "birth" rate of the capital in the neoclassical framework. Second, for both models a closed-form solution exists if the key differential equation is a *Bernoulli* differential equation. Third, for the present examples, the asymptotic speed of convergence features the constant population birth rate in the Malthusian setting and the constant "death" rate of the capital intensity in the neoclassical framework.

2 Malthus - A Closed-Form Solution

2.1 Model and Steady State

At t the economy is endowed with an aggregate production function

$$Y(t) = A\, T^{\alpha}\, N(t)^{1-\alpha},\ 0 < \alpha < 1, \tag{1}$$

where $Y(t)$ denotes output, $A > 0$ the level of the technology, $T > 0$ the available amount of arable land, and $N(t)$ the population. Accordingly, per-capita income is

$$y(t) \equiv \frac{Y(t)}{N(t)} = A \left(\frac{T}{N(t)} \right)^{\alpha}. \tag{2}$$

It increases with the level of the technology and the land intensity. Let $n(t) \equiv \dot{N}(t)/N(t)$ denote the population growth rate and assume that n is monotonically increasing in y according to

$$n(t) = c_b - \frac{c_d}{y(t)}\,, \text{ with } c_b,\, c_d > 0. \tag{3}$$

I henceforth associate c_b with the birth rate and $c_d/y(t)$ with the death rate. Upon combining (2) and (3) one obtains

$$\frac{\dot{N}(t)}{N(t)} = c_b - \frac{c_d}{A}\left(\frac{T}{N(t)}\right)^{-\alpha}. \tag{4}$$

It captures the Malthusian feedback between population growth and the means of subsistence, which, in turn, depend on current population (see, Malthus (1798)). It is useful to characterize the steady state defined as any pair $(\bar{N}, \bar{y})$ that satisfies $n(t) = 0$.

Lemma 1 *There is a unique and globally stable steady state with*

$$\bar{y} = \frac{c_d}{c_b} \text{ and } \bar{N} = T\left[\frac{A}{\bar{y}}\right]^{\frac{1}{\alpha}}. \tag{5}$$

Proof Equation (5) is immediate from (4) for $n(t) = 0$. Global stability follows as (4) implies a negative relationship between $n(t)$ and $N(t)$. Thus, for $N(t) \gtrless \bar{N}(t)$ we have $n(t) \lesseqgtr 0$. ■

At the steady state per-capita income is at the level of Malthusian subsistence, $\bar{y}$, and population is constant. From (2) and (4) we deduce that decreasing returns to population are the driving force behind Lemma 1: a larger population than $\bar{N}$ implies a lower land intensity, lower per-capita income than $\bar{y}$, and, in turn, a higher death rate. For a constant birth rate, population growth is negative and population declines.

It is interesting to compare this mechanism to the neoclassical growth model. Using the notation of Barro and Sala-í-Martin (2004), p. 43, we have

$$\frac{\dot{k}(t)}{k(t)} = s\,A\,k(t)^{-(1-\alpha)} - (n + d), \tag{6}$$

which corresponds to (4). Here, $k(t) \equiv K(t)/N(t)$ is the capital intensity at t, $s > 0$ the savings rate, $A > 0$ a technology parameter, n the exogenous population growth rate, $d > 0$ the depreciation rate, and α the output elasticity of capital in the underlying Cobb-Douglas production function $Y(t) = A\,K(t)^{\alpha}\,N(t)^{1-\alpha}$.

The growth rate of the capital intensity can be interpreted as the difference between a birth rate and a death rate. The birth rate is gross investment, i. e. aggregate savings per unit of current capital. Decreasing returns to capital force it to decline as $k(t)$ increases. The death rate is the sum of two constants. The capital intensity literally dies because of depreciation. As a per-capita magnitude, it also "dies" because new workers are born. Thus, contrary to the Malthusian model, here decreasing returns affect the birth rate rather than the death rate and generate a unique and globally stable steady state.

2.2 Paths of Population and Per-Capita Income

Equation (4) is a Bernoulli differential equation, which allows us to express the evolution of population and per-capita income in closed-form.[1] To see this, write (4) as

$$\dot{N}(t)\,N(t)^{-(1+\alpha)} = c_b\,N(t)^{-\alpha} - \frac{c_d}{A\,T^{1-\alpha}} \tag{7}$$

and define $v(t) \equiv N(t)^{-\alpha}$. Then, (7) can be expressed as

$$-\frac{\dot{v}(t)}{\alpha} = c_b\,v(t) - \frac{c_d}{A\,T^{\alpha}} \tag{8}$$

or, with (5),

$$\dot{v}(t) = -\alpha\,c_b\,v(t) + \alpha\,c_b\,\bar{N}^{-\alpha}$$

This is a first-order ordinary differential equation with constant coefficients. The solution to the initial value problem, where $v(0) = v_0 = N_0^{-\alpha}$, is[2]

$$v(t) \equiv N(t)^{-\alpha} = (N_0^{-\alpha} - \bar{N}^{-\alpha})\,e^{-\alpha\,c_b\,t} + \bar{N}^{-\alpha}. \tag{9}$$

[1] Bernoulli equations are non-linear differential equations of the form $\dot{z(t)} + a(t)\,z(t) = b(t)\,z(t)^r$, where r is any real number different from 0 and +1 (see, e. g. , Gandolfo (1997)). In view of (4), we have $z(t) := N(t)$, $a(t) := -c_b$, $b(t) := -c_d/(A\,T^{\alpha})$, and $r := 1+\alpha$.

[2] To highlight the structural similarity to the neoclassical model let $v(t) \equiv k(t)^{1-\alpha}$. Then, (6) can be written as

$$\frac{\dot{v}(t)}{1-\alpha} = -(n+d)\,v(t) + s\,A.$$

It follows that

$$v(t) = (k_0^{1-\alpha} - \bar{k}^{1-\alpha})\,e^{-(1-\alpha)\,(n+d)\,t} + \bar{k}^{1-\alpha},$$

where $\bar{k}$ is the steady-state level of $k(t)$. These expressions correspond to (8) and (9), respectively.

Thus, the gap between $N(t)^{-\alpha}$ and its steady state value vanishes at the constant rate $\alpha\, c_b$. Following some straightforward manipulations based on (9), we obtain

Proposition 1 *The paths of* $N(t)$ *and* $y(t)$ *are*

$$N(t) = \left(\frac{\bar{N}^{\alpha}}{1 - \left(1 - \left(\frac{\bar{N}}{N_0}\right)^{\alpha}\right) e^{-\alpha\, c_b\, t}} \right)^{\frac{1}{\alpha}}$$

and

$$y(t) = \bar{y} + (y_0 - \bar{y})\; e^{-\alpha\, c_b\, t}.$$

Thus, if $N_0 \gtrless \bar{N}$ and therefore $y_0 \lessgtr \bar{y}$, population declines (increases) and per-capita income increases (declines) over time. The steady-state values of (5) appear as the limits for $t \to \infty$.

The rate $\alpha\, c_b$ is the approximate speed of convergence in the neighborhood of the steady state. It measures by how much $n(t)$ declines as population size increases in a proportional sense. Denote β the speed of convergence, then

$$\beta(t) \equiv -\frac{\partial n(t)}{\partial \ln N(t)}.$$

To determine $\beta(t)$, $\ln N(t)$ has to appear in (4), i. e. $n(t) = c_b - (c_d/\,(A\,T^{\alpha}))\; e^{\alpha\, \ln N(t)}$. We infer that

$$\beta(t) = \alpha\, \frac{c_d}{A} \left(\frac{N(t)}{T}\right)^{\alpha} = \alpha\, c_b \left(\frac{N(t)}{\bar{N}}\right)^{\alpha},$$

where the second step uses (5). As $N(t) \to \bar{N}$ for $t \to \infty$ it follows that

$$\bar{\beta} \equiv \lim_{t\to\infty} \beta = \alpha\, c_b. \tag{10}$$

Similarly, the asymptotic speed of convergence associated with y is[3]

$$\lim_{t\to\infty} \beta_y(t) \equiv \lim_{t\to\infty} -\frac{\partial(\dot{y}(t)/y(t))}{\partial \ln y(t)} = \bar{\beta}. \tag{11}$$

Hence, asymptotically, the convergence coefficient for n is the same as that for y. From (10) it is equal to the elasticity of per-capita income with respect to the

[3] To see this, use (2) and (3) to find that $\dot{y}(t)/y(t) = -\alpha\, n(t) = -\bar{\beta} + \alpha\, c_d\, e^{-\ln y(t)}$. Thus, $\beta_y(t) = \alpha\, c_d/y(t) = \bar{\beta}\, \bar{y}/y(t)$. Taking the limit for $t \to \infty$ gives (11).

land intensity, α, times the constant birth rate, c_b. The analogous result for the neoclassical model of (6) is that the convergence coefficients for k coincides with the one for y. Both are equal to the elasticity of the output-capital ratio with respect to the capital intensity, $1 - \alpha$, times the constant "death" rate of the capital intensity, $n + d$.

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