E C O N O M I C S   B U L L E T I N

Multitask Rank Order Tournaments

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Abstract

This work extends Lazear and Rosen's seminal paper to evaluate the performance of rank order tournaments when agents perform multiple tasks and the principal chooses, together with the prize spread, the weights assigned to each task in determining aggregate performance of each agent. All essential results of one−dimensional tournaments generalize to a multi−dimensional setting. However, the relative performance of tournaments and linear piece rates is shown to also depend on the covariance between measurement errors.

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1 Introduction

A tournament is a compensation scheme based purely on the ranking of agents' performance. In their seminal contribution, Lazear and Rosen [7] compare tournaments and a linear piece rate mechanism and conclude that, when agents are risk neutral, both incentive schemes are capable of achieving a first-best outcome, while, with risk averse agents, the optimal mechanism depends, amongst other factors, on the structure of the errors in performance measurement. In a subsequent paper, Green and Stokey [1] generalize Lazear and Rosen work, showing that if agents are subject to both idiosyncratic and common stochastic influences, then an optimal tournament dominates optimal independent contracts if the distribution of the common shock is sufficiently diffuse.

Although the emphasis of tournament theory is in organizational economics[1], applications can be found in other areas as well.

For instance, Govindasamy et.al. [2] suggest the use of tournaments as an environmental policy tool to address nonpoint-source pollution problems. Furthermore, some first attempts can be observed towards the introduction of tournaments as regulatory instruments; this seems to be the case, for example, if we look at the way chosen by OFWAT to provide incentives to water utilities for improvements in their standard of service[2].

As tournament theory seems to be fairly developed, it is quite surprising that no attempt exists to investigate the properties of rank-based mechanisms in a multitask setting[3]. This is particularly relevant because it is seldom the case that agents' duties involve just a single task. The performance of workers in a firm, for example, could be related both to the amount of output each of them produces and to the quality of each piece produced.

The aim of this paper is to take a first step towards a full investigation of the properties of rank order tournaments when more than one task is performed by agents. We will, therefore, follow [7] and compare the efficiency properties of a rank based payment scheme with those of a linear "piece rate" mechanism, where the payment made by the principal to the agents has the form of a "wage" per unit of performance.

Of course, with multiple dimensions in the agents' performance, we lose one of the advantages of tournaments, namely the reduced informational needs as compared to those of other incentive mechanisms: the principal has, in fact, to observe absolute performances of each agent on *each* task, to understand who is the winner of the "contest". However, other desirable theoretical properties of tournaments extend to a multi-task setting. First, tournaments are capable of "filtering out" random events that are common across agents (see Lazear [6]). Second, tournaments have a "fixed-slot" structure, so that the principal does not gain by lying *ex post* about the performance of the agents. Following Malcomson [8], this is particularly relevant in situations where agents cannot verify the signal their supervisor receives concerning their output. On the other hand, a tournament specifies prizes that do not vary with the *ex-post* observation of agents' outcomes, and, therefore, can be thought of as a solution to the principal's commitment problem[4].

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[1] For a survey see, for example, the dedicated part in [11].

[2] See [10].

[3] For a review of incentives and mechanism design problems under multiple tasks see, for example, [4], [12] and the literature cited therein.

[4] Holmstrom [3] shows, in a single task setting, that a tournament can be an "informationally wasteful" incentive scheme; a contract indexing compensation to individual performance and to an average of peers'

The paper is organized as follows. First, Section 2 presents the main features of the model. Second, Section 3 is central in the paper, as we show that it is possible to design tournaments in a multi-task setting. Third, in Section 4, we compare the incentive properties of tournaments with those of a linear compensation scheme as the one proposed by Holmstrom and Milgrom [4]. Finally, section 5 concludes.

2 The Model

We analyze the "effort" choice problem of two agents (for example two workers in a firm or two regulated firms) that perform $j = 1, .., n$ tasks on behalf of a principal (inside a firm, that could be a hierarchical superior of the two workers; in the regulatory example, it could be an environmental protection agency or a utilities regulator). These efforts generate, on one hand, gross benefits to the principal, and, on the other hand, a vector of information signals.

The signal the principal receives concerning task j for each agent i ($i = 1, 2$) (call it α_{ij}) is determined by the effort a_{ij} performed by each agent, by one idiosyncratic random term ε_{ij} and by one error term that is common across agents η_j; analytically[5]:

$$\alpha_{ij} = a_{ij} + \varepsilon_{ij} + \eta_j$$

To simplify notation (in everything that follows, vectors are represented by boldface letters), $\boldsymbol{a_i} = (a_{i1} \dots a_{in})$, $\boldsymbol{\alpha_i} = (\alpha_{i1} \dots \alpha_{in})$ and $\boldsymbol{\epsilon_i} = (\epsilon_{i1} \dots \epsilon_{in})$, $\boldsymbol{\eta} = (\eta_1 \dots \eta_n)$. Clearly, $E(\boldsymbol{\alpha_i}|\boldsymbol{a_i}) = \boldsymbol{a_i}$.

The two agents receive a *net* private benefit from exerting effort; the corresponding benefit function, $b_p(.)$, is assumed to be identical across agents, strictly concave and twice continuously differentiable. This formulation allows for multiple interpretations. In the regulatory example, $b_p(.)$ is the regulated firms' profit function. On the other hand, when agents are workers who have no intrinsic motivation for their work, then $b_p(.) = -c(.)$, where $c(.)$ is the private cost of undertaking effort.

The random vectors, $\boldsymbol{\epsilon_i}$ and η, are assumed to be distributed normally and identically across agents, with 0 mean and covariance matrix, respectively, Σ_ϵ and Σ_η. We also assume that the idiosyncratic error terms are independently distributed across agents, and that $\boldsymbol{\epsilon_i}$ and η are independent random vectors. The sum of the idiosyncratic and common error terms is, therefore, distributed as a normal law, with mean 0 and covariance matrix $\Sigma = \Sigma_\epsilon + \Sigma_\eta$.

Agent i receives a payment $w_i(.)$ that can depend on the vector of all possible signals (thus, including the signals related to the tasks performed by the other agent).

We assume that each agent's expected utility function takes the form: $E\left(u\left[w_i\left(.\right) + b_p(.)\right]\right)$, where $u(x) = -e^{-\rho x}$ and ρ measures the agents' constant (absolute) risk aversion.

The principal is risk neutral and has a very simple objective function, based on the assumption that he is interested only in the effort input of the agents:

performance would be a better way to rule out common shocks. On the other hand, as [6] points out, such a relative performance scheme would lose the "commitment advantage" discussed in the text.

[5] Apparently, we assume thus a somewhat simpler relation between the effort levels and the generated signals than Holmstrom and Milgrom. However, as Holmstrom and Milgrom themselves point out in footnote 8, this is just a matter of how we define the agents' choice variables.

$$\sum_{i=1}^{2} \left(B\left(\boldsymbol{a_i}\right) - E(w_i) \right), \tag{1}$$

where $B\left(\boldsymbol{a_i}\right)$ are expected gross benefits and $E(w_i)$ is the expected payment made by the principal to agent i. We do not impose *a priori* restrictions on the sign of the payment. Note that this formulation uses the implicit assumption that there are no externalities between the effort levels undertaken by the two agents - relaxing this assumption constitutes an obvious area for further research.

3 Tournaments

We shall now turn to a two-players tournament in which the rules of the game specify a fixed prize W_1 to the winner and a fixed prize W_2 to the loser, so that $W_1 > W_2$.

Furthermore, before the "game" takes place, the principal commits to the weights given to each outcome in determining the "overall performance" of each player; call γ_j the weight on task j.

"Overall observed performance" for player i will, therefore, be $\boldsymbol{\gamma\alpha_i}^T$ (where $\boldsymbol{\gamma} = (\gamma_1 \ldots \gamma_n)$), and the winner will be defined as the agent that gets the highest value for this indicator.

Our aim is to determine the competitive prize structure (described by the spread among the winner's and the loser's wage, $W = W_1 - W_2$) and a set of weights γ_j. To do this, we first derive effort strategies by the two agents, given two arbitrary prizes' values and an arbitrary level for the $\gamma_j s$.

The two risk averse agents maximize their expected utility.

Agent 1's expected wage is $\overline{W} \equiv PW_1 + (1-P)W_2$, where P is the probability of winning for the same agent.

The variance of agent 1's wage is: $P(W_1 - \overline{W})^2 + (1 - P)(W_2 - \overline{W})^2$, which can be simplified to: $(W_1 - W_2)^2(1 - P)P$.

Therefore, the certainty equivalent of agent 1's expected utility is[6]:

$$PW_1 + (1 - P)W_2 + b_p(.) - \frac{1}{2}\,\rho\,(W_1 - W_2)^2\,(1 - P)\,P. \tag{2}$$

Agent 1's probability of winning is given by: $P = prob\left[\boldsymbol{\gamma\alpha_1}^T > \boldsymbol{\gamma\alpha_2}^T\right]$, or $P = G\left[\boldsymbol{\gamma}\left(\boldsymbol{a_1} - \boldsymbol{a_2}\right)^T\right]$, where G is the cumulative distribution function of the "composite error term", given by $\boldsymbol{\gamma}\left(\boldsymbol{\epsilon_2} - \boldsymbol{\epsilon_1}\right)^T$ [7].

Thus, as pointed out in the introduction, tournaments are capable of sorting out common error terms in the multitask case exactly as they do in the single task case.

If we obtain an interior solution for all effort levels (thus, $a_{1j} > 0$ for all j), then agent 1's FOCs w.r.t. a_{1j} are: $\frac{\partial P}{\partial a_{1j}}\,(W_1 - W_2) + \frac{\partial b_p(.)}{\partial a_{1j}} - \frac{1}{2}\,\rho\,(W_1 - W_2)^2\,(1 - 2P)\,\frac{\partial P}{\partial a_{1j}} = 0$, or

[6] The approximation used in the text for the agents' certainty equivalent follows Milgrom and Roberts [9, p. 246-247]. This approximation is valid, in particular, if the variance of the agents' income is not "too high", and/or for a sufficiently low value of the risk aversion coefficient ρ.

[7] The assumptions with respect to the idiosyncratic errors imply that the composite error term is normally distributed with mean 0 and variance $\sigma_\Sigma^2 = 2\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T$.

(where $g(.)$ is the (normal) probability density function of the "composite error term" and where $W = W_1 - W_2$):

$$g(.)W\gamma_j + \frac{\partial b_p(.)}{\partial a_{1j}} - \frac{1}{2}\,\rho\,W^2\,(1-2P)\,g(.)\,\gamma_j = 0. \tag{3}$$

Following the same steps, as the two agents are *ex ante* identical, it is easy to show that agent 2's FOCs are symmetric.

We focus, then, on a symmetric Nash equilibrium: $a_{1j} = a_{2j} = a_j$. If such an equilibrium exists[8], then $P = G(0) = \frac{1}{2}$.

Let $\boldsymbol{b'_p} = \left(\frac{\partial b_p(.)}{\partial a_{i1}} \dots \frac{\partial b_p(.)}{\partial a_{in}}\right)$. The agents' FOCs can then be written as follows (taking into account that $g(0) = \frac{1}{\sqrt{2\pi\sigma_\Sigma^2}}$):

$$\frac{1}{2\sqrt{\pi\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T}}\,W\boldsymbol{\gamma} = -\boldsymbol{b'_p}, \tag{4}$$

Under a tournament, the principal maximizes: $\left(\sum_{i=1}^{2} B\left(\boldsymbol{a_i}\right)\right) - W_1 - W_2$, under two constraints: 1) the agents must receive a non negative expected utility; 2) the agents' FOCs must be satisfied. From (4) it is clear that the amount of effort that is performed by the two agents will only depend on the spread between the two payments. Then, we assume that W_2 is used to satisfy the agents' participation constraint; given symmetry, the solution of the principal's problem implies the maximization of the total certainty equivalent per agent[9]:

$$B\left(\boldsymbol{a}\right) + b_p\left(\boldsymbol{a}\right) - \frac{1}{8}\,\rho\,W^2. \tag{5}$$

under incentive compatibility constraints given in (4). This leads us to the first important result with tournaments[10] (for the proof, see Appendix A):

Proposition 1 *Under a tournament:*

- *the optimal prize spread is given by:* $W = 2\sqrt{\pi\boldsymbol{b'_p}\Sigma_\epsilon\left(\boldsymbol{b'_p}\right)^T}$*;*
- *the optimal effort vector is defined by*[11]*:* $\left(\rho\,\pi\,[B_{kj}]\,\Sigma_\epsilon - I\right)^{-1}\left(\boldsymbol{B'}\right)^T = \boldsymbol{b'_p}^T$.

This optimal effort vector can be implemented using a set of weights that satisfies $\boldsymbol{\gamma} = -\,\boldsymbol{b'_p}$[12].

[8] The existence of a pure strategy *symmetric* Nash equilibrium will be assumed in what follows. Further, following the essential single-task tournaments literature, we will focus our analysis on such an equilibrium, although the symmetry of the two agents' FOCs does not imply *per se* that a symmetric Nash equilibrium - if it exists - is unique.

[9] We leave out the index i wherever there is no ambiguity.

[10] We shall assume here that the second order conditions for the principal's problem are satisfied.

[11] We call $\left[B_{kj}\right]$ the $n \times n$ matrix with kj element $\frac{\partial^2 b_p(.)}{\partial a_j \partial a_k}$.

[12] Notice that $\boldsymbol{\gamma} = -\boldsymbol{b'_p}$ implies negative weights for the tasks that generate net marginal benefits for the agents in equilibrium. This is rather intuitive: agents that are intrinsically motivated in performing a task $\boldsymbol{j}$ in equilibrium would be willing to *expand* their effort on that task beyond the optimal level; the "correct" incentives imply therefore that the effort on such task is "taxed".

From Proposition 1 it is clear that tournaments allow to obtain first-best effort levels when agents are risk-neutral; as a consequence, a first essential result in tournaments theory has been shown to be robust to our multi-task extension.

4 A comparison with linear piece rates

Following Holmstrom and Milgrom [4, p. 29]), suppose that the linear compensation scheme takes the form $s+\boldsymbol{r\alpha_i}^T$, where $\boldsymbol{r} \equiv (r_1 \dots r_j \dots r_n)$. The fixed wage s will be used to allocate total surplus between the principal and the agents, and that the optimal effort vector is given by the following condition:

$$(\rho\,[B_{kj}]\,\Sigma - I)^{-1}\,(\boldsymbol{B'})^T = \left(\boldsymbol{b'_p}\right)^T = -\boldsymbol{r}^T. \tag{6}$$

where $[B_{kj}]$ has the same meaning as in Proposition 1[13].

Let us now compare this with the results we have obtained with tournaments.

Substituting the optimal prize spread from Proposition 1 in (5) we get the maximized value of the total certainty equivalent per agent under tournaments, that is $CE_t(\boldsymbol{a_t}) = B\,(\boldsymbol{a_t}) + b_p\,(\boldsymbol{a_t}) - \frac{\pi}{2}\,\rho\,\boldsymbol{b'_p}\,(\boldsymbol{a_t})\ \Sigma_\epsilon\ \left(\boldsymbol{b'_p}\,(\boldsymbol{a_t})\right)^T$, where $\boldsymbol{a_t}$ is the corresponding optimal effort vector. Suppose now that the principal introduces a linear piece rate system that induces the same effort levels. The corresponding total certainty equivalent per agent is: $CE_{pr}(\boldsymbol{a_t}) = B\,(\boldsymbol{a_t}) + b_p\,(\boldsymbol{a_t}) - \frac{1}{2}\,\rho\,\boldsymbol{b'_p}\,(\boldsymbol{a_t})\ \Sigma\ \left(\boldsymbol{b'_p}\,(\boldsymbol{a_t})\right)^T$. This yields:

$$CE_{pr}(\boldsymbol{a_t}) - CE_t(\boldsymbol{a_t}) = \frac{\pi}{2}\,\rho\,\boldsymbol{b'_p}\,(\boldsymbol{a_t})\ \Sigma_\epsilon\ \boldsymbol{b'_p}\,(\boldsymbol{a_t})^T - \frac{1}{2}\,\rho\,\boldsymbol{b'_p}\,(\boldsymbol{a_t})\ \Sigma\ \boldsymbol{b'_p}\,(\boldsymbol{a_t})^T \tag{7}$$

We then obtain immediately (compare with Proposition 1 and Corollary 1 in Green and Stokey [1]):

Proposition 2 *Suppose there is no common shock. There always exists a linear piece rate contract that is Pareto-superior to the optimal tournament.*

Proof. If there is no common shock, then $\Sigma_\epsilon = \Sigma$. The proof is completed by noting that $\pi > 1$. ∎

Now consider the optimal linear piece rate scheme; call $\boldsymbol{a_p}$ the corresponding effort vector satisfying (6). Then, consider a tournament that achieves the same effort levels[14].

If we evaluate (7) at $\boldsymbol{a_p}$, we get the difference between the maximized total certainty equivalent under optimal linear piece rates and the total certainty equivalent under a tournament that induces the same effort vector. Thus, we obtain:

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[13] Condition (6) corresponds to Equation (5) in Holmstrom and Milgrom [4, p. 32]. Notice that, coherently with what happens in the tournaments case, the conditions that define efficient linear piece rates contracts require a negative value of r_j for the tasks that generate positive net marginal benefits for the agents in equilibrium.

[14] In order to induce the effort vector $\boldsymbol{a_p}$ with tournaments, it is enough to set $W = 2\sqrt{\pi\ \boldsymbol{b'_p}\,(\boldsymbol{a_p})\ \Sigma_\varepsilon\ \left(\boldsymbol{b'_p}\,(\boldsymbol{a_p})\right)^T}$ and $\boldsymbol{\gamma} = -\boldsymbol{b'_p}\,(\boldsymbol{a_p})$.

Proposition 3 *Suppose that the idiosyncratic terms and the common shocks are independent. There exists a tournament that dominates the optimal linear piece rates contract if:*

- *there is at least one task for which the variance of the common error term is high enough,*
- *or if there is at least one couple of tasks for which the positive covariance in the common error terms is high enough and, from* both *tasks, the agents obtain negative net marginal benefits (or positive net marginal benefits), evaluated at the effort levels induced by the optimal linear piece rates contract,*
- *or if there is at least one couple of tasks for which the negative covariance in the common error terms is high enough in absolute value, and the net marginal benefits obtained by agents have opposite signs (again, evaluated at the effort levels induced by the optimal linear piece rates contract).*

This argument is independent of the agents' degree of risk aversion.

Proof. Under the assumption that the idiosyncratic terms and the common shocks are independent, the covariance matrix of the sum of the two error terms is simply the sum of the individual covariance matrices: $\Sigma = \Sigma_\epsilon + \Sigma_\eta$. Just note then that: $\boldsymbol{b'_p}\Sigma_\eta \left(\boldsymbol{b'_p}\right)^T = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_p(.)}{\partial a_j}\frac{\partial b_p(.)}{\partial a_k}\sigma_{jk}^{\eta}$. ■

The first part of this Proposition generalizes a known result (see [7]) to a multi-task setting. On the other hand, the second and third part are completely new and it can be worthwhile thinking about their interpretation.

The case when the second part of Proposition 3 is more likely to apply in "real life" is when the agents perform tasks that only bring costs to them (for example, two tedious jobs from which no job satisfaction is derived). The third part, instead, considers couples of tasks that yield marginal net benefits with opposite signs. Here, an example we might think of is environmental protection. In the social optimum, a firm's marginal private benefit of increasing production will typically be positive, while the marginal private benefit of pollution abatement will be negative.

Let us now look at the common error vector - following Lazear and Rosen [7, p. 856-857], two interpretations can be given.

First, it can refer to a set of "activity-specific measurement errors". For instance, consider the case of two agents performing two tasks on behalf of a firm; if the two tasks are measured by a common monitoring device (or supervisor), and this monitoring device is the same for both agents, a high covariance between "common" measurement errors is likely to arise. In this case, tournaments are preferable to a linear piece rate scheme if the agents derive negative (or positive) private marginal benefits from both tasks.

Second, each common error can refer to a "true random variation" affecting all agents in the same way. For instance, we can think of salesmen who sell two products whose sales are affected by negatively correlated common shocks (sunglasses and umbrellas for instance) - in that case, our model suggests that tournaments are dominated by piece rates if net marginal benefits, deriving from different selling efforts, have the same sign.

5 Conclusion

This work extends Lazear and Rosen [7] paper on rank order tournaments to a multitask setting where the choice variables of the principal are the prize spread and the weights given to each task in determining aggregate performance of each agent. Our first conclusion is that many essential results of one-dimensional tournaments are robust to a multi-dimensional setting extension. We also provide fundamentally new insights in that we show how covariance between error terms affects the relative performance of tournaments and linear piece rates when agents perform multiple tasks on behalf of their principal.

Of course, this paper is just a first step. Possible generalizations follow directly from some restrictive assumptions we have used: first, further insights could be gained from relaxing the assumption of identical net benefits function for the agents (note that Lazear and Rosen [7] analyze this problem with *risk-neutral* agents); second, it could be worthwhile generalizing the setting to an arbitrary number of agents (as in Green and Stokey [1]).

References

[1] Green, J.R., Stokey, N.L. (1983), "A Comparison of Tournaments and Contracts", *Journal of Political Economy* 91, 349-364.

[2] Govindasamy, R., Herriges, J.A., Shogren, J.F. (1994), "Nonpoint Tournaments", in C. Dosi and T. Tomasi, *Nonpoint Source Pollution Regulation: Issues and Analysis*, Kluwer Academic Publishers, Dordrecht.

[3] Holmstrom, B. (1982), "Moral Hazard in Teams", *Bell Journal of Economics* 13, 324-340.

[4] Holmstrom, B., Milgrom, P. (1991), "Multitask Principal Agent Analyses: Incentive Contracts, Asset Ownership and Job Design", *Journal of Law, Economics and Organization* 7, 24-52

[5] Gibbons, R., Waldman, M. (1999), "Careers in Organizations", in O. Ashenfelter and D. Card, *Handbook of Labor Economics*, Volume 3, Elsevier.

[6] Lazear, E.P. (1995), *Personnel Economics*, MIT Press, Cambridge, Massachussets.

[7] Lazear, E.P., Rosen, S. (1981), "Rank Order Tournaments as Optimum Labor Contracts", *Journal of Political Economy* 89, 841-864.

[8] Malcomson, J.M. (1984), "Work Incentives, Hierarchy, and Internal Labor Markets", *Journal of Political Economy* 92, 486-507

[9] Milgrom, P., Roberts, J. (1992), *Economics, Organization and Management*, Prentice Hall, Englewood Cliffs.

[10] OFWAT (1999), "Future water and sewerage charges 2000 - 05". http://www.ofwat.gov.uk/

[11] Prendergast, C. (1999), “The Provision of Incentives in Firms”, *Journal of Economic Literature* 37, 7-63.

[12] Sinclair-Desgagne, B. (1999), “How to Restore Higher-Powered Incentives in Multitask Agencies”, *Journal of Law, Economics and Organization* 15, 418-433.

A Proof of Proposition 1

The Lagrangian per agent is:

$$B(\boldsymbol{a}) + b_p(\boldsymbol{a}) - \frac{1}{8}\rho W^2 + \boldsymbol{\lambda}\left(W\boldsymbol{\gamma} + 2\sqrt{\pi\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T}\boldsymbol{b_p'}(\boldsymbol{a})\right)^T, \tag{8}$$

where $\boldsymbol{\lambda} \equiv (\lambda_1 \ldots \lambda_n)$ is the vector of Lagrange multipliers associated with the vector of constraints (4).

The FOCs with respect to $\boldsymbol{a}$ yield:

$$(\boldsymbol{B'})^T + \left(\boldsymbol{b_p'}\right)^T + 2\sqrt{\pi\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T}\,[B_{kj}]\,(\boldsymbol{\lambda})^T = 0, \tag{9}$$

where $\boldsymbol{B'} = \left(\frac{\partial B(.)}{\partial a_{i1}} \ldots \frac{\partial B(.)}{\partial a_{in}}\right)$ and $[B_{kj}]$ is the $n \times n$ matrix with kj element $\frac{\partial^2 b_p(.)}{\partial a_j \partial a_k}$.

If $[B_{kj}]$ is non-singular, then we can solve this for $\boldsymbol{\lambda}^T$ (using $\left(2\sqrt{\pi\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T}\right)^{-1} = g(0)$):

$$\boldsymbol{\lambda}^T = -g(0)\,[B_{kj}]^{-1}\left(\boldsymbol{B'} + \boldsymbol{b_p'}\right)^T. \tag{10}$$

The FOCs with respect to $\boldsymbol{\gamma}$ yield:

$$W\boldsymbol{\lambda}^T + \left(\boldsymbol{b_p'}\boldsymbol{\lambda}^T\right)\frac{1}{\sqrt{\pi\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T}}2\pi\Sigma_\epsilon\boldsymbol{\gamma}^T = 0. \tag{11}$$

From (4), we see that: $\boldsymbol{b_p'}\boldsymbol{\lambda}^T = -\frac{1}{2\sqrt{\pi\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T}}\,W\left(\boldsymbol{\gamma}\boldsymbol{\lambda}^T\right)$. Substitute this and $(g(0))^2 = \frac{1}{4\pi\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T}$ in (11):

$$\left(\boldsymbol{\lambda}^T - 4\pi\,(g(0))^2\left(\boldsymbol{\gamma}\boldsymbol{\lambda}^T\right)\Sigma_\epsilon\boldsymbol{\gamma}^T\right)W = 0. \tag{12}$$

If the principal is willing to provide any effort incentives, then $W > 0$ and (12) can only be satisfied if (substitute (10)):

$$-g(0)\,[B_{kj}]^{-1}\left(\boldsymbol{B'} + \boldsymbol{b_p'}\right)^T = 4\pi\,(g(0))^2\left(\boldsymbol{\gamma}\boldsymbol{\lambda}^T\right)\Sigma_\epsilon\boldsymbol{\gamma}^T. \tag{13}$$

The FOC for W yields:

$$\boldsymbol{\lambda}\boldsymbol{\gamma}^T = \frac{1}{4}\rho W. \tag{14}$$

Substitute (14) in (13) (remember that $\boldsymbol{\lambda}\boldsymbol{\gamma}^T$ is a scalar and therefore equal to its transpose) and divide both sides by $g(0)$:

$$-\left[B_{kj}\right]^{-1}\left(\boldsymbol{B'}+\boldsymbol{b'_p}\right)^T = \pi g(0)\rho\, W\Sigma_\epsilon\boldsymbol{\gamma}^T. \tag{15}$$

Substitute (4) in (15):

$$-\left[B_{kj}\right]^{-1}\left(\boldsymbol{B'}+\boldsymbol{b'_p}\right)^T = -\pi\;\rho\;\Sigma_\epsilon\boldsymbol{b'_p}^T. \tag{16}$$

This can be re-arranged further to:

$$\left(\rho\;\pi\left[B_{kj}\right]\Sigma_\epsilon - I\right)^{-1}\left(\boldsymbol{B'}\right)^T = \boldsymbol{b'_p}^T. \tag{17}$$

(17) gives us the effort levels.

In order to find W, note that (4) can be rewritten as $\boldsymbol{\gamma} = -\frac{1}{g(0)\,W}\boldsymbol{b'_p}$. Substitute this in (14) to obtain:

$$-\frac{1}{g(0)\;W}\boldsymbol{\lambda}\left(\boldsymbol{b'_p}\right)^T = \frac{1}{4}\,\rho\,W. \tag{18}$$

Combining (16) with (10) yields $\boldsymbol{\lambda} = -g(0)\;\rho\;\pi\boldsymbol{b'_p}\Sigma_\epsilon$, and therefore (18) leads to:

$$2\sqrt{\pi\boldsymbol{b'_p}\Sigma_\epsilon\left(\boldsymbol{b'_p}\right)^T} = W. \tag{19}$$

As Σ_ϵ is a covariance matrix and therefore nonnegative definite, there exists a real solution for W.

Finally, in order to find $\boldsymbol{\gamma}$, substitute (19) in (4):

$$\sqrt{\boldsymbol{b'_p}\Sigma_\epsilon\left(\boldsymbol{b'_p}\right)^T}\,\boldsymbol{\gamma} = -\sqrt{\boldsymbol{\gamma}\Sigma_\epsilon\boldsymbol{\gamma}^T}\,\boldsymbol{b'_p}. \tag{20}$$

which of course holds for $\boldsymbol{\gamma} = -\boldsymbol{b'_p}$.