E C O N O M I C S   B U L L E T I N

Bootstrap inference on Fully Modified Estimates of Cointegrating Coefficients: A Comment

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Abstract

A bootstrap algorithm proposed by Psaradakis (2001) for hypothesis testing in I(1) regressions is discussed and shown to be valid only under the null hypothesis. A simple correction making the procedure valid under both the null and the alternative hypothesis is proposed.

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1 Introduction

The widespread awareness of the poor small sample performances of asymptotic tests in regressions with integrated variables is leading to a growing interest in simulation-based inference. In this vein, in a recent contribution to this journal Psaradakis (2001) has proposed a bootstrap algorithm for hypothesis testing in fully modified ordinary least squares (FMOLS; Phillips and Hansen, 1990) and canonical cointegrating regressions (CCR; Park,1992). The aim of this note is to show (Section 2) that under the alternative hypothesis the proposed bootstrap algorithm is not valid and to propose a simple correction making the algorithm valid under both the null and the alternative hypothesis. A small Monte Carlo experiment, discussed in Section 3, suggests that the proposed bootstrap algorithm has size close to nominal significance level and good power properties.

2 Bootstrap hypothesis testing in I(1) regression

Consider for simplicity, and with no loss of generality, a cointegrated bivariate system

$$\begin{aligned} y_t &= \beta x_t + u_{1t} \\ x_t &= x_{t-1} + u_{2t} \end{aligned} \tag{1}$$

where $t = 1, \ldots, n$ and $\{y_t\}_{t=1}^n$ $\{x_t\}_{t=1}^n$are both $I(1)$. The first equation is a cointegrating relation, and thus $\{u_{1t}\ ,\ u_{2t}\}_{t=2}^n$ is stationary. Suppose we are interested in testing some hypothesis $H_0 : \beta = \beta_0$. If β is estimated by FMOLS or CCR inference is asymptotically standard, but the small sample properties of such tests may be poor, with Type I errors much higher than the nominal significance levels. Psaradakis (2001) proposed to tackle this important problem by applying a bootstrap test, with the pseudoseries constructed according to the following approach:

- compute the residuals of the cointegrating regression under H_0, $\widehat{u}_{1t}^0 = y_t - \beta_0 x_t$, as well as the first differences of the right hand side variable $\widehat{u}_{2t} = \Delta x_t$;

then, there are essentially two alternatives:

(a) first option (suggested): *sieve boostrap*:

1. fit AR models to $\{\widehat{u}_{1t}^{0}\}_{t=1}^{n}$ and $\{\widehat{u}_{2t}\}_{t=1}^{n}$ and obtain residuals $\{\widehat{e}_{1t}^{0}\}_{t=p+1}^{n}$ and $\{\widehat{e}_{2t}\}_{t=p+1}^{n}$, where p is the order of the model, selected by some consistent criteria such as AIC;

2. resample the residuals of the AR models computed in step (1), obtaining $\{\widehat{e}_{1t}^{0*}, \widehat{e}_{2t}^{*}\}_{t=p+1}^{n}$

3. construct recursively first the series of bootstrap errors $\{u_{1t}^{*}, u_{2t}^{*}\}_{t=1}^{n}$ on the basis of the estimated coefficients of the AR models and the resampled residuals $\{\widehat{e}_{1t}^{0*}, \widehat{e}_{2t}^{*}\}_{t=p+1}^{n}$, and finally the bootstrap pseudoseries $\left\{y_t^* = \beta_0 x_t + u_{1t}^*,\ x_t^* = x_{t-1}^* + u_2^*\right\}_{t=1}^{n}$ (initial values set at zero in both recursions).

(b) second option (examined, but not suggested): *block bootstrap*: resample directly $\{\widehat{u}_{1t}^{0}, \widehat{u}_{2t}\}_{t=1}^{n}$ and proceed to construct y_t^* and x_t^* as above.

Some simulation results for the Type I errors are reported by Psaradakis: the test based on the sieve bootstrap delivers the best results; the block bootstrap performance is disappointing for all block lengths, with the resulting test strongly conservative; the tendency to overreject of the asymptotic test is confirmed. According to Psaradakis, the algorithm is justified by the common assumption (henceforth *AR representation assumption*) that the noise $\mathbf{u}_t = [u_{1t}\ u_{2t}]'$ admits an AR(∞) representation $\mathbf{u}_t = \sum_j^\infty \Pi_j \mathbf{u}_{t-j} + \epsilon_t$, with ϵ a white-noise process and the coefficient matrices Π_j satisfying $det\left(I_m - \sum_j^\infty \Pi_j z^j\right) \neq 0\ \ \forall\ z \leq 1$. Thus, in practice $\mathbf{u}_t$ may be filtered by a finite-order AR model to obtain IID residuals to be resampled, as in the sieve bootstrap, or directly resampled applying a scheme allowing for weak dependence, such as the block bootstrap. However, the algorithm as outlined by Psaradakis is *not* applied to the unconstrained residuals $\mathbf{u}_t$, but to the residuals $\{\widehat{u}_{1t}^{0},\ \widehat{u}_{2t}\}_{t=2}^{n}$ estimated under the null hypothesis $H_0 : \beta = \beta_0$. Now, these satisfy the AR representation assumption if, and only if, H_0 holds: if this does not happen (i.e., the alternative hypothesis holds) $\widehat{u}_1^0$ will be non-stationary, and thus $\widehat{\mathbf{u}}_t^0 = \sum_j^\infty \Pi_j \widehat{\mathbf{u}}_{t-j}^0 + \varepsilon_t$ will have a unit root. This will have serious consequences:

(*i*) the block bootstrap, which cannot handle non-stationarity (Paparoditis and Politis, 2003), would fail;

(*ii*) the sieve bootstrap will empirically work if, and only if, the unit root in the finite-order AR model fitted to $\{\widehat{u}_{1t}^{0}\}_{t=1}^{n}$is precisely estimated .

Can we devise a correction to the Psaradakis procedure which will be make it valid under both the null and the alternative hypothesis? Fortunately, the answer is positive and very simple. Considering that the bootstrap pseudoseries should resemble the real data while satisfying $H_0 : \beta = \beta_0$, the natural solution is to apply the sieve bootstrap to the *unconstrained* residuals $\left\{\widehat{u}_{1t} = y_t - \widehat{\beta}x,\ \widehat{u}_2 = \Delta x_t\right\}_{t=2}^{n}$, where $\widehat{\beta}$ is some suitable estimator (e.g., FMOLS or CCR) of the cointegrating coefficient β, and then proceed exactly as above. If the system is cointegrated $\{\widehat{u}_{1t}\ , \widehat{u}_{2t}\}_{t=2}^{n}$ will be stationary, the AR representation assumption will be satisfied, and both the sieve bootstrap and the block bootstrap may be applied. This approach (actually easier to implement as the estimation of the constrained model is not required) is proposed by Psaradakis for the construction of confidence intervals, while for hypothesis testing is recommended by van Giersbergen and Kiviet (2002) in stable dynamic regression models and by Omtzigt and Fachin (2002) in the Johansen framework. Having established a theoretically valid scheme we need to shed some light on its empirical performance: this will be done in the next section.

3 Simulation study

We examined through a very small Monte Carlo experiment the performance of a sieve bootstrap $t-$test based on pseudodata constructed according to both the Psaradakis algorithm (based upon the constrained residuals) and the corrected version proposed here (based upon the unconstrained residuals). Given that the block bootstrap is known not to be valid for non-stationary data we did not take it into consideration. In order to facilitate comparisons the experimental design replicated Psaradakis'; however, we considered only two points in the parameter space, namely the most and least favourable to his algorithm. As we will see, the results are clear enough to suggest that it was unnecessary to replicate the experiment in all the other cases, thus limiting the amount of space required to report the results. Further, we limited the analysis to the FMOLS estimator since replicating the study for the test based upon the CCR estimator would not provide any additional information useful for the comparison of the two bootstrap procedures.

The data generating process (DGP) is common to many studies, starting with Phillips and Hansen (1990), and is given by system (1) with errors governed by

$$\begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} + \begin{bmatrix} 0.3 & -0.4 \\ \theta & 0.6 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \end{bmatrix} \tag{2}$$

where

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \sim NID\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \sigma \\ \sigma & 1 \end{bmatrix} \right). \tag{3}$$

The DGP parameters are as follows: $\beta = 2$; $(\theta, \sigma) = (-0.4, -0.5)$, $(0.8, 0.5)$, respectively the combination where the Psaradakis procedure delivers the best and worst results under H_0. The sample size is 50, with 30 initial observations discarded; the number of Monte Carlo replications 1000, with 399 bootstrap redrawings. The lag length of the AR models fitted by OLS to the residuals is selected by the AIC, with a maximum lag equal to 3. Finally, to evaluate power we considered the wrong null hypothesis $H_0 : \beta = 3$. Given that we are interested in the relative comparison of the performance of the two tests the choice of this specific value was dictated by the need to have an hypothesis not too close to the DGP value, so to obtain constrained residuals with a clear non-stationary pattern even in a sample as small as that considered here.

The results are summarised in Table 1. As it can be appreciated by looking at either the reported rejection rates for the traditional 5% and 10% tests or the Kolmogorov- Smirnov test for the hypothesis that the $p-$values are uniform over [0,1], when H_0 holds the procedure based on the unconstrained residuals works as well as the Psaradakis procedure. The rejection rates are not significantly different and the KS statistics are always very close and largely smaller than the 5% critical value (0.043). When H_0 does not hold the unit root in the constrained residuals is on the average rather accurately estimated, and thus the Psaradakis procedure delivers a good performance (actually, maximum power, a result due to the extremely high signal/noise ratio) which is fully matched by the corrected procedure.

What can we learn from this small simulation? First of all, as anticipated the Psaradakis procedure may actually work well, even if the null hypothesis does not hold, if the unit root in the residuals is precisely estimated. However, since the seminal work of Dickey and Fuller (1979) we know this to be a difficult task; thus, the good performance of the Psaradakis procedure in our Monte Carlo experiment should be considered the exception (probably explained by the very simple dynamics of the DGP) rather than the rule. The second result of interest is that the procedure based on the unconstrained residuals works well under both H_0 (when using the constrained residuals would obviously be more efficient) and H_1. Thus, there is absolutely no reason to prefer the risky option of using the constrained residuals. Replacing these with the unconstrained residuals we obtain a bootstrap algorithm which:

(*i*) is theoretically valid both under the null and the alternative hypothesis;

(*ii*) not requiring estimation of the constrained model, is actually simpler to implement;

(*iii*) is empirically able to deliver good small sample performances.

and which is thus the recommended approach.

Table 1
Bootstrap t-tests in FMOLS regressions

	H_0 *true*				H_0 *false*			
θ, σ	$[-0.4, -0.5]$		$[0.8, 0.5]$		$[-0.4, -0.5]$		$[0.8, 0.5]$	
$Test$	SB_0	SB_U	SB_0	SB_U	SB_0	SB_U	SB_0	SB_U
KS	0.016	0.017	0.08	0.07	–	–	–	–
$\sum \widehat{\pi}$	0.13	0.08	-0.12	-0.12	0.99	0.22	0.98	-0.24
R_5	4.40	5.10	8.40	9.50	100.0	100.0	100.0	100.0
R_{10}	9.90	9.50	15.10	15.40	100.0	100.0	100.0	100.0

SB_0: Sieve bootstrap test with constrained residuals;
SB_U: Sieve bootstrap test with unconstrained residuals;
KS: test for H_0 : rejection rates uniform over [0,1];
$\sum \widehat{\pi}$: average sum of the estimated autoregressive polynomial;
R_α: Rejection rate of test with nominal significance level=α;
$-$: not relevant.

4 References

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