E C O N O M I C S   B U L L E T I N

Matching Markets: the Particular Case of Couples

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Abstract

We consider one−to−one markets where workers can apply to a centralized procedure either alone or as a couple, in order to reach a core− stable matching of firms and workers. We impose stringent restrictions on the preferences of couples capturing the fact that spouses want to live in the same region, which are shown to be incompatible with the guarantee of the existence of a stable matching.

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1 Introduction

According to Checker (1973) the proportion of students who look for positions through the National Resident Matching Program (N.R.M.P.) has been decreasing since the mid 1970s. Roth and Sotomayor (1990) attributes this problem to an increase in the number of couples among students. Their explanation to the phenomenon is that couples were discouraged from participating in the N.R.M.P. because this mechanism did not properly take into account their preferences for finding jobs in the same geographical area. In March 1998, a new algorithm for the N.R.M.P. was adopted (see Roth and Peranson (1999) for a detailed description of the mechanism) which did take into account couples' preference for proximity. Unfortunately it may fail to guarantee the existence of a stable matching. Indeed, Roth (1984) presented an example of non-existence when couples have preferences over both, their own and their partner's matches. The example, however, relies on preferences which seem very hard to motivate intuitively, and certainly do not seem to be related in any obvious way to the notion of geographical proximity.

This paper analyses the existence of stable matchings when couples' concern for proximity is taken into consideration. In contrast to Roth (1984), we restrict couples' preferences so as to capture the desire of couples to live nearby each other by partitioning positions into 'regions', and assuming that couples prefer to be matched in the same region rather than in distinct ones. We assume further that couples share a ranking of regions. Though one might have hoped this additional structure would solve the non- existence problem, this turn out not to be the case. We present a relatively straightforward example of a matching market in which couples' preferences display this structure and which nevertheless fails to have a stable matching.

There have been other attempts at dealing with the problem of couples. Dutta and Massó (1997) is one such work. However, their domain of preferences is certainly too restrictive, since they practically assume that couples behave like single people. Klaus and Klijn (2003) defines a responsiveness condition (the preferences of couples are responsive if the unilateral improvement of one partner is beneficial to both members), and show that, under this condition, the existence of a stable matching is guaranteed. Nevertheless, responsiveness does not capture adequately situations in which employment for both partners within the same region is the first priority for a couple. Klaus, Klijn and Massó (2003) shows that, under responsiveness, couples may manipulate the 1998 procedure by applying as single workers, rather than as couples. They also show that, under responsiveness, the 1998 procedure may fail to find a stable matching (even when one exists).

In Section 2 we introduce the notation and describe the restrictions on couples' preferences. In Sections 3, we present our example of non- existence and we remark on our result in Section 4.

2 Preliminaries

2.1 Agents, preferences, and matchings

A *one-to-one matching market with couples* is a quadruple $(F, W, C, \succ)$ where F and W are two disjoint and finite sets. The set of *firms* is $F = \{f_1, ..., f_r\}$ and $W = \{w_1, ..., w_m\}$ is the set of *workers*. Generic firms and workers will be denoted by f and w respectively. Assume that some workers are married, yielding a fixed number n of *couples*. Denote by C the set of such couples. Therefore, we can partition the set of workers between the set of single workers, denoted by W_s, and the set of married workers, denoted by W_c. Given the subset W_c we can look at the set of couples C as a particular subset of $W_c \times W_c$. Each firm f has a strict, transitive and complete preference relation $\succ_f$ over $W \cup \{\emptyset\}$. We interpret the empty set in the preference relation $\succ_f$ as firm f not being assigned to any worker. When a firm f ranks the empty set above a worker w, it means that it does not want to hire w. Single workers have a strict, transitive and complete preference relation $\succ_w$ over the set $F \cup \{\emptyset\}$. We interpret the empty set in $\succ_w$ as w being unemployed. The preferences of members of a couple are defined on her/his match and those of her/his spouse, i.e., on $(F \cup \{\emptyset\}) \times (F \cup \{\emptyset\})$. Let (w_1, w_2) be a couple, $(f_1, f_2) \succ_{w_1} (f_3, f_4)$ means that worker w_1 prefers being matched to f_1 and her partner, w_2, to f_2, than being matched to f_3 and her partner to f_4. Preference profiles are $(r + m)$-tuples of preference relations and they are represented by $\succ = (\succ_{f_1}, ..., \succ_{f_r}, \succ_{w_1}, ..., \succ_{w_m})$. From now on we assume as given a particular one-to-one matching market with couples $(F, W_s, C, \succ)$.

For any firm f we define $A_f(\succ)$, the *acceptable set of f under $\succ$*, to be the set of workers strictly preferred to the empty set. We represent the preference relation of a firm as an ordered list of its acceptable workers. Likewise we define $A_w(\succ)$ for any $w \in W_s$, the *acceptable set of w under $\succ$*. For any couple $(w_1, w_2) \in C$ we define the *acceptable set of (w_1, w_2) under $\succ$* to be the set of pairs (firms or the empty set) preferred to the empty set by both spouses, it is denoted $A_{w_1,w_2}(\succ)$. Still we do not impose any restriction on the preferences of couples, in particular partners may have different orders, nevertheless we assume that there is no pair preferred to the empty set by one member of a couple which is not preferred to the empty set by the other member.

A *matching* μ is a mapping from the set $F \cup W$ into $F \cup W \cup \{\emptyset\}$ such that for all $f \in F$ and $w \in W$: (1) either $|\mu(f)| = 1$ and $\mu(f) \in W$, or $\mu(f) = \emptyset$, (2) either $|\mu(w)| = 1$ and $\mu(w) \in F$, or $\mu(w) = \emptyset$, (3) $\mu(w) = f$ if and only if $w = \mu(f)$.

2.2 Stability of matching markets with couples

Given a matching μ we say that firm f *blocks* μ if it prefers remaining alone to being matched to $\mu(f)$; i.e., $\emptyset \succ_f \mu(f)$. A single worker w *blocks* μ if $\emptyset \succ_w \mu(w)$. Finally the matching μ is *blocked by a couple* $(w_1, w_2) \in C$ if either:

a- $[(\emptyset, \mu(w_2)) \succ_{w_1} (\mu(w_1), \mu(w_2))$ and $(\mu(w_2), \emptyset) \succ_{w_2} (\mu(w_2), \mu(w_1))]$,

b- $[(\mu(w_1), \emptyset) \succ_{w_1} (\mu(w_1), \mu(w_2))$ and $(\emptyset, \mu(w_1)) \succ_{w_2} (\mu(w_2), \mu(w_1))]$, or

c- $[(\emptyset, \emptyset) \succ_{w_1} (\mu(w_1), \mu(w_2))$ and $(\emptyset, \emptyset) \succ_{w_2} (\mu(w_2), \mu(w_1))]$.

A matching is *individually rational* if it is not blocked by any individual firm, any single worker or any couple.

A matching μ is *blocked by a pair* (w, f) where $w \in W_s$, if $w \neq \mu(f)$, $w \succ_f \mu(f)$ and $f \succ_w \mu(w)$. In our focus, we consider that a couple blocks a matching only if partners agree to do so, namely we do not consider instability due to the decision making within couples. Thus, a matching μ is *blocked by a couple* (w_1, w_2) if either:
a- only one member blocks

$$\begin{array}{c} w_1 \neq \mu(f), w_1 \succ_f \mu(f),\ (f, \mu(w_2)) \succ_{w_1} (\mu(w_1), \mu(w_2)) \\ \text{and } (\mu(w_2), f) \succ_{w_2} (\mu(w_2), \mu(w_1)), \end{array}$$

b- one member blocks the matching, the other one quits

$$\begin{array}{c} w_1 \neq \mu(f), w_1 \succ_f \mu(f),\ (f, \emptyset) \succ_{w_1} (\mu(w_1), \mu(w_2)) \\ \text{and } (\emptyset, f) \succ_{w_2} (\mu(w_2), \mu(w_1)), \text{ or} \end{array}$$

c- both members block simultaneously

$$w_1 \neq \mu(f_1), w_2 \neq \mu(f_2), w_1 \succ_{f_1} \mu(f_1), w_2 \succ_{f_2} \mu(f_2),$$
$$(f_1, f_2) \succ_{w_1} (\mu(w_1), \mu(w_2)) \text{ and } (f_2, f_1) \succ_{w_2} (\mu(w_2), \mu(w_1)).$$

We say that a matching μ is group-wise stable if it is not blocked by any pair or any couple.

Definition 1 *A matching is* <u>*stable*</u> *if it is individually rational and group-wise stable.*

2.3 Restriction on preferences of couples

We define for couples the *strong regional lexicographic* domain of preferences. We refer to F^c as the *partition* of F for couple c. Condition (1) bellow indicates that there exists a common ranking of the elements of their partition of the set of firms, where two firms belongs to the same element when there are located in the same region. Furthermore, couples have preferences on acceptable pairs of firms that depend on the element of the partition they belong to, we denote most preferred elements with a lower index. Condition (2) says that couples always prefer being matched to the same element of the partition, independently of its desirability, to situations where spouses are separated or only one of them has a position; we call it *togetherness.*

Definition 2 *Let $c = (w_1, w_2) \in C$. We say that orderings $\succ_{w_1}$ and $\succ_{w_2}$ satisfy the* <u>*strong regional lexicographic conditions*</u> *if*

1- there exists a partition of F, $F^c = \{F_1^c, ..., F_p^c\}$, such that for all pairs of firms $(f_1, f_2) \in F_q^c \times F_q^c$ and $(f_3, f_4) \in F_{q'}^c \times F_{q'}^c$, $q \neq q'$, both being acceptable for (w_1, w_2),

$$\text{if } q < q' \text{ then } [(f_1, f_2) \succ_{w_1} (f_3, f_4) \text{ and } (f_2, f_1) \succ_{w_2} (f_4, f_3)]; \qquad (1)$$

2- for all pairs of firms $(f_1, f_2) \in F_q^c \times F_q^c$, being acceptable for (w_1, w_2), then for all pairs of partners $p_1 \in (F_{q'}^c \cup \{\emptyset\})$, $p_2 \in (F_{q''}^c \cup \{\emptyset\})$, $q' \neq q''$, we have

$$(f_1, f_2) \succ_{w_1} (p_1, p_2) \text{ and } (f_2, f_1) \succ_{w_2} (p_2, p_1). \qquad (2)$$

Until now the partitions of couples are unrelated whereas their existence is motivated by geographical restrictions, which should be similar for all couples. Therefore, we impose that all couples share the same partition but not necessarily the same ordering of elements.

Definition 3 *Let* C *be the set of couples and assume that their preferences satisfy the strong regional lexicographic conditions. We say that couples face the* <u>*same geographical constraint*</u> *if there is a partition* $\{F_1, ..., F_p\}$ *of* F *such that for all couples* $c \in C$*, there exists a one-to-one permutation* $\sigma_c : \{1, ..., p\} \to \{1, ..., p\}$ *with the property that* $\{F_1^c, ..., F_p^c\} = \{F_{\sigma_c(1)}, ..., F_{\sigma_c(p)}\}$.

3 Independence of hiring and existence of stable matchings

3.1 No guarantee in general

We defined stringent restrictions on the preferences of couples. Nevertheless they are not sufficient to guarantee the existence of a stable matching.

Theorem 1 *Let* $(F, W_s, C, \succ)$ *be a one-to-one matching market with couples and assume that preferences of couples satisfy the strong regional lexicographic conditions and that couples face the same geographical constraint. It may be that no stable matching exists.*

Example 1 shows the negative result.

Example 1 Let $(F, W, C, \succ)$ be the one-to-one matching market with couples where $F = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$, $W_s = \emptyset$, $c_1 = (w_1, w_2)$, $c_2 = (w_3, w_4)$, $c_3 = (w_5, w_6)$, and $\succ$ is such that

$\succ_{f_1}$	$\succ_{f_2}$	$\succ_{f_3}$	$\succ_{f_4}$	$\succ_{f_5}$	$\succ_{f_6}$
w_1	w_5	w_3	w_3	w_1	w_5
w_4	w_2	w_6	w_6	w_4	w_2

and all workers are acceptable to f_7. Assume that couples face the same geographical constraint and that their ranking of the "regions" is the same, namely $F_1^c = \{f_1, f_2, f_3\}$, $F_2^c = \{f_4, f_5, f_6, f_7\}$ for any $c = c_1, c_2, c_3$; moreover, assume $\succ_{w_1}$, $\succ_{w_2}$, $\succ_{w_3}$, $\succ_{w_4}$, $\succ_{w_5}$ and $\succ_{w_6}$ satisfy togetherness. Finally all firms are acceptable to all workers, f_7 being unanimously considered as the worst firm. We argue in the Appendix that, in Example 1, there is no stable matching.

The result also holds when the number of firms and workers is the same: if one extends the previous example introducing a single agent w_7 least preferred by f_7 and unacceptable to other firms, still there is no stable matching.

3.2 Substitutability and togetherness

It is worth comparing the role of togetherness in our model to the one of substitutability in many-to-one markets. There, firms have preferences over subsets of workers. If no restriction is put on the preferences of firms, the following may happen: at a given matching, a firm f with one free position desires worker w_1 instead of w_2 because she complements well another match, w_3. Then (f, w_1) is a blocking pair while (f, w_2) is not. Nevertheless if f is matched to another worker w_4 and now, w_2 complements w_4 well, (f, w_2) is a blocking pair while (f, w_1) is not. In other words these blocking pairs (hence stability) depend on other match(es) of the firm; if this happens it may be that no stable matching exists. Kelso and Crawford (1982) defines, for general matching markets with salary negotiation, the gross substitute condition, adapted to our setting by Roth (1984) as the restriction of substitutability, which gets rid of this dependence and guarantees the existence of a stable matching. In Cantala (2004) we formalize independence by establishing that the Choice of firms has the property of Independence of Irrelevant Alternative.

Togetherness is the key assumption in our model; it both characterizes the preferences of couples and breaks the guarantee of stability. The intuition is the following: suppose that a couple (w_1, w_2) is matched in a region to f_1 and f_2 respectively; and that w_1 is the best choice for f_1 while w_2 is not for f_2. Then f_2 may wish to block the matching in particular with a worker w_3 belonging to couple (w_3, w_4); nevertheless the decision of w_3 to block the matching with f_2 depends on the match of w_4. If w_4 does get a position then w_2 is fired by f_2 and if w_2 cannot get any other position in the neighborhood, w_1 may decide to quit her position to look for a position together with w_2 elsewhere. That is to say, one may observe a domino effect: the dismissal of w_2 leads w_1 to quit f_1. If a cyclical sequence of domino effects occurs, there may be no stable matching in the market, as in Example 1. The domino effects are due to the fact that the attractiveness of a firm for a member of a couple depends on the match of her spouse. This dependence generates instability in the market and is due to togetherness, therefore it is a pervasive feature of markets with couples.

3.3 Alternative approaches

Consider the urban area formed by Boston, New York City and Philadelphia. It is likely that couples do not accept that one spouse works in Boston while the other works in Philadelphia, even if the area is attractive to the couple. To mimic the preferences of couples, the mechanism should ask both the ordering of the regions and the maximal distance between firms the couple would accept. Note that in the New York City area, this maximal distance may be smaller than the distance between two cities; specifically the above mentioned spouses may agree to one of them working in New York City and the other in Boston or Philadelphia, but would reject a situation where one partner works in Boston and the other in Philadelphia. Is this extra information helpful? Indeed it is not, in Example 1 it is sufficient to interpret the two regions as Boston and Philadelphia.

Example 1 also extends to a network setting when a couple prefers any matching where partners are matched to two firms connected by a link, to any other situation, namely matchings where they are matched to unconnected firms or one of them is unmatched, otherwise preferences of partners are independent.

4 Final remark

Roth and Parenson (1999) perform what they call "theoretical computations" so as to investigate to what extend theoretical failures affect the practical outcome of the N.R.M.P.. They observe, in particular, that the probability for the set of Core stable matchings to be empty becomes very small when markets become large. Our analysis does not provide a definitive answer to why the issue of couples becomes less problematic when markets get increasingly large. Our modeling of regions, however, provides the following conjecture: since the instability of the market is due to the fact that couples 'move' from one region to another so as to look for positions nearby, when the number of firms increases, since the number of existing region is independent of the size of the market[1], the number of firms increases in each region; hence couples will have to move less frequently in order to look for positions together, and the phenomenon becomes less important.

[1] It results of geographical criteria.

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6 Appendix

Proof that in Example 1 there is no stable matching.
Consider, without loss of generality (w_3, w_4). The types of individually rational matchings are:
TYPE 1 (w_3) is unmatched.
Since w_3 prefers being matched to remaining alone, (w_3, f_4) and (w_3, f_3) are two blocking pairs.
TYPE 2 (w_3) is matched to f_7.
Then (w_3, f_4) is a blocking pair, whatever the rest of the matching is.
TYPE 3 (w_3) is matched to f_3.
Case 1: (w_4) is matched to f_1, (w_5) is matched to f_2
3.1.1. if w_6 is matched to f_4 or f_7, (w_5, f_6) is a blocking pair by togetherness,
3.1.2 otherwise $((w_5, w_6), (f_6, f_4))$ is a blocking quadruple by togetherness.
Case 2: (w_4) is matched to f_1, (w_5) is not matched to f_2
3.2.1 if w_2 is matched to f_2, (w_1, f_1) is a blocking pair by togetherness,
3.2.2 otherwise $((w_1, w_2), (f_1, f_2))$ is a blocking quadruple by togetherness.
TYPE 4 (w_3) is matched to f_4.
Case 1: (w_4) is matched to f_5 and (w_1) is matched to f_1
4.1.1 if w_5 is not matched to f_2, (w_5, f_2) is a blocking pair if w_6 is matched to f_3, otherwise by togetherness $((w_5, w_6), (f_2, f_3))$ is a blocking quadruple.
4.1.2 if w_5 is matched to f_2, (w_1, f_5) is a blocking pair if w_2 is matched to f_6 or f_7, otherwise by togetherness $((w_1, w_2), (f_5, f_6))$ is a blocking quadruple.
Case 2: (w_4) is matched to f_5 and (w_1) is not matched to f_1
then $((w_3, w_4), (f_3, f_1))$ is a blocking quadruple since they prefer the region

and are still together.
Case 3: (w_4) is not matched to f_5
4.3.1 if w_1 is matched to f_1, (w_4, f_5) is a blocking pair by togetherness,
4.3.2 if w_1 is not matched to f_1, $((w_3, w_4), (f_3, f_1))$ is a blocking quadruple since they prefer the region, and also according to togetherness.
Therefore, there is no stable matching.■

In markets where preferences are only defined over the other side of the market (firms over workers or groups of workers and workers over firms for instance), it is well known that it exists a stable matching unanimously most-preferred by workers and least-preferred by firms, and a stable matching unanimously most-preferred by firms and least-preferred by workers, as established first by Gale and Shapley (1962) for one-to-one markets, Crawford and Knoer (1981) for general markets and in a dynamic setting by Kelso and Crawford (1982) and Crawford (1991) in particular. On the contrary, Aldershof and Carducci (1996) shows that there is no such unanimity among firms and among workers, nor this sharp opposition of interest between both sides.[2] Example 2 shows this under our restriction of preferences.

Example 2 Let the market $(F,\ W_s,\ C,\ \succ)$ such that $F = \{f_1, f_2, f_3, f_4\}$, $W_s = \emptyset$, $(w_1, w_2) = c_1$, $(w_3, w_4) = c_2$ and

$$\begin{array}{rl} \succ_{f_1} = \succ_{f_3} = & \{w_1\}\ \{w_4\}, \\ \succ_{f_2} = \succ_{f_4} = & \{w_3\}\ \{w_2\}, \end{array}$$

while

$$\begin{array}{rl} \succ_{w_1} = & \{(f_1, f_2)\}\ \{(f_3, f_4)\}\ ..., \\ \succ_{w_2} = & \{(f_2, f_1)\}\ \{(f_4, f_3)\}\ ..., \\ \succ_{w_3} = & \{(f_2, f_1)\}\ \{(f_4, f_3)\}\ ..., \\ \succ_{w_4} = & \{(f_1, f_2)\}\ \{(f_3, f_4)\}\ ...; \end{array}$$

where f_1 and f_2 form a 'regions', as well as f_3 and f_4. With these preferences, there are two stable matchings

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_2 & w_4 & w_3 \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_4 & w_3 & w_1 & w_2 \end{pmatrix}.$$

Notice that firm f_1 and f_3 and workers w_1 and w_2 prefer μ to μ', while firms f_2 and f_4 and workers w_3 and w_4 prefer μ' to μ.

[2] Analysis of the lattice structure of the set of stable matchings can be found in Blair (1988), Alkan (2001), Martinez, Massó, Neme and Oviedo (2001) and Roth and Sotomayor (1990) for a review.