E C O N O M I C S B U L L E T I N

"On the differentiability of the benefit function": correction and addendum

Jean−Michel Courtault
Universite de Franche−Comte, Libre

Bertrand Crettez
Universite de Franche−Comte, Libre

Nayla Hayek
Universite de Franche−Comte, Libre, Cermsem

Abstract

The proposition 1 of our paper "On the differentiability of the benefit function" (Economics Bulletin, March, 24) is incorrect under the assumption 1. We provide two alternative versions of this assumption under which the statement of proposition 1 holds true.

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1. Introduction

This note provides a correction to Proposition 1 of our paper, "On the differentiability of the benefit function", Economics Bulletin, March 2004, page 2. Indeed, the statement of Proposition 1 does not follow from the assumption H1 used in the paper, which is not correctly set. We propose two alternative versions of assumption H1 under which Proposition 1 is true. The first version amounts to a very small change in the statement (the proof of proposition 1 being almost unchanged). The second version enables us to prove differentiability under a new and rather weak assumption (moreover, in the course of the proof, we establish the continuity of the benefit function at the point under study without assuming that g is good, as it is done in Luenberger (1992)).

2. Correction and addendum

We now propose two new versions of assumption 1 which will enable us to prove the differentiability of the benefit function.

First version

(H1) *We assume that:* $x_0 \in R^n_{++}$ *and* $x_0 - b(x_0, \alpha_0)g \in R^n_{++}$. *We also suppose that* $\langle \nabla U(x_0 - b(x_0, \alpha_0)g), g \rangle \neq 0$ *and that* g *is l-good at* x_0, *i.e.: there is a neighborhood* W_{x_0} *of* x_0 *such that for all* x *in* W_{x_0}, *for all* β_1, β_2, *if* $\beta_1 > \beta_2$, *then* $U(x + \beta_1 g) > U(x + \beta_2 g)$.

Under this hypothesis, the proof of Proposition 1 is unchanged except the following line (line 23). We have $\alpha = U(x - \psi(x, \alpha)g) > U(x - b(x, \alpha)g)$ since g is l-good by assumption. This contradicts the fact that $U(x - b(x, \alpha)g) \geq \alpha$.

The difference between this assumption and that used in the paper, is that in the later only positive increases in the direction g generate increases in preferences. Notice that the old assumption H1 implies a local version of the new one.

Second version

We consider now a different hypothesis H1. Its feature is similar to that of the assumption H1 used in the paper. The new proof makes use of the continuity of the benefit function at point (x_0, α_0). Luenberger (1992), Proposition 4, page 465, shows the continuity of the benefit function at each point where it is finite valued under the hypotheses that $U(.)$ is continuous and g is good everywhere, i.e. : for all x, for all positive α, $U(x + \alpha g) > U(x)$. Our (very simple) proof of the continuity of $b(.,.)$ at point (x_0, α_0) uses the assumptions that make it possible to apply the implicit function theorem (i.e. continuous differentiability of the utility function $U(.)$, regularity and interiority conditions). There is no need to assume that g is good. Assuming that g is locally good at $x_0 - b(x_0, \alpha_0)g$ in the following sense given in H1 will finally yield the differentiability of the benefit function.

(H1) *We assume that:* $x_0 \in R^n_{++}$ *and* $x_0 - b(x_0, \alpha_0)g \in R^n_{++}$. *We also suppose that* $\langle \nabla U(x_0 - b(x_0, \alpha_0)g), g\rangle \neq 0$ *and that* g *is locally good at* $x_0 - b(x_0, \alpha_0)g$, *i.e.: there is a neighborhood* W *of* $x_0 - b(x_0, \alpha_0)g$, *there is a positive real number* m *such that for all* z *in* W, *for all positive* β *smaller than* m *one has:* $U(z + \beta g) > U(z)$.

Under this hypothesis we remove W_{x_0} from line 9 of the proof of Proposition 1. The rest is unchanged till line 23 where we add the following after $b(x, \alpha) \geq \psi(x, \alpha)$: We show now that $b(.,.)$ is continuous at (x_0, α_0). Since $U(.)$ is continuous , $b(.,.)$ is upper semi continuous on $V_{x_0} \times V_{\alpha_0}$ (wherever it is finite); see Luenberger (1992), proposition 3, page 464. Let $(x_n, \alpha_n)_n$ be a sequence such that for every n, $(x_n, \alpha_n) \in V_{x_0} \times V_{\alpha_0}$ and the sequence converges to (x_0, α_0). Since $b(x_n, \alpha_n) \geq \psi(x_n, \alpha_n)$ we have $\liminf_{n \to +\infty} b(x_n, \alpha_n) \geq \liminf_{n \to +\infty} \psi(x_n, \alpha_n) = \psi(x_0, \alpha_0) = b(x_0, \alpha_0)$. Hence $b(.,.)$ is lower semi continuous at (x_0, α_0), so it is continuous at (x_0, α_0). Therefore there exists a neighborhood N_{x_0} of x_0, a neighborhood N_{α_0} of α_0 such that for every $(x, \alpha) \in V_{x_0} \times V_{\alpha_0} \cap N_{x_0} \times N_{\alpha_0}$, $x - b(x, \alpha)g \in W$. Moreover, since the function $(b - \psi)(.,.)$ is continuous at (x_0, α_0), there exists a neighborhood N'_{x_0} of x_0, a neighborhood $N'_{\alpha_0}$of α_0 such that for every $(x, \alpha) \in V_{x_0} \times V_{\alpha_0} \cap N'_{x_0} \times N'_{\alpha_0}$, $|b(x, \alpha) - \psi(x, \alpha) - (b(x_0, \alpha_0) - \psi(x_0, \alpha_0))| \leq m$, i.e., $|b(x, \alpha) - \psi(x, \alpha)| \leq m$ since $b(x_0, \alpha_0) = \psi(x_0, \alpha_0)$. Hence, taking the intersection of all these neighborhoods yields a neighborhood of (x_0, α_0) that we denote again $V_{x_0} \times V_{\alpha_0}$ for simplification. So let $(x, \alpha) \in V_{x_0} \times V_{\alpha_0}$. Suppose $b(x, \alpha) > \psi(x, \alpha)$. The end of the proof is the same as in the paper.

Reference

Courtault, J.M., Crettez, B. and N. Hayek, 2004, "On the Diffentiability of The Benefit Function", *Economics Bulletin*, March, 24.

Luenberger, D.G., 1992 a, "Benefit Functions and Duality", *Journal of Mathematical Economics*, 21, 461–481.