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Good and bad objects: the symmetric difference rule

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Abstract

We consider the problem of ranking sets of objects, the members of which are mutually compatible. Assuming that each object is either good or bad, we axiomatically characterize a cardinality−based rule which arises naturally in this dichotomous setting.

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1 Introduction

In this paper we consider the problem of ranking sets of objects the members of which are mutually compatible. Situations for which such a ranking can be of use include matching, choice of assemblies, election of new members of a committee, group identification and coalition formation. Very often in these situations the a priori information one has about a decision maker's opinion is simple: it takes the form of a partition of the set of all objects into "good" and "bad" items.

For example, if the set of objects is a set of individuals, the set of good objects can consist of all individuals who share a given property or who are qualified for a job (cf. Kasher and Rubinstein (1997), Samet and Schmeidler (2003)). Accordingly, in this case the set of bad objects will include all individuals who do not have this property or who are not qualified. Or, in a slightly different context, the set of good objects can contain all candidates who deserve to become new society members according to the opinion of some society founder (cf. Berga et al. (2003)), while the set of bad objects will consist of all candidates who do not deserve this grace. Notice that the specification of good and bad objects in these contexts implies homogeneity and full substitutability of the objects within a particular group.

Given such a dichotomous setting, the question we ask in the present paper is the following: how can one extend in a meaningful way this rudimentary information about one's opinion over the single objects to an ordering on the power set of the set of objects? We answer this question by presenting an axiomatic characterization of what we call the *symmetric difference* rule. According to this rule, a set of objects C is considered to be better than another set of objects D if and only if the cardinality of the symmetric difference between C and the set of all good objects is smaller than the cardinality of the symmetric difference between D and the set of all good objects.

It is not difficult to see that the symmetric difference rule induces a unique additive separable preference relation over the set of all groups of objects, the set of all good objects being its top and the set of all bad objects being its bottom. Having this in mind, our result can be interpreted as a characterization of a subclass of the class of separable preferences, the latter being commonly used as a primitive in the analysis of voting situations (cf. Barberà et al. (1991), Ju (2003)) and coalition formation games (cf. Burani and Zwicker (2003)). On the other hand, this paper contributes to the problem of ranking sets of objects in general. Especially, it can be seen as a translation of the purely cardinality-based freedom ranking of opportunity sets (cf. Pattanaik and Xu (1990)) to the context of ranking sets of objects the members of which are mutually compatible (cf. Barberà et al. (2003)).

2 Definitions and axioms

Throughout the paper, let X be a nonempty finite set of objects. These objects may be for example candidates considered for membership in a club, possible coalitional partners, bills considered for adoption by a legislature, etc. We assume that each object is either good or bad, and that there is at least one good object and at least one bad object. We denote by G the set of all good objects in X, and by B the set of all bad objects in X. Notice that $B = X \setminus G$.

In what follows, the set of all subsets of X will be denoted by $\mathcal{X}$. The elements of $\mathcal{X}$ are the (alternative) groups of objects an agent may be confronted with. The question arises how this agent ranks sets consisting on different numbers of good and bad items based on his partition $\{G, B\}$ of X. Consequently, the problem to be analyzed in this paper is how to establish a reflexive, transitive and complete binary relation $\succeq$ on $\mathcal{X}$. For all $C, D \in \mathcal{X}$, $C \succeq D$ is to be interpreted as "C is at least as good as D". The asymmetric and symmetric factors of $\succeq$ will be denoted by $\succ$ ("is better than") and $\sim$ ("is as good as"), respectively.

Let $\mathcal{P}$ be the set of all reflexive, transitive and complete binary relations on $\mathcal{X}$. As an element of $\mathcal{P}$, we define the binary relation $\succeq_{\triangle}$ based on the *symmetric difference* with G as follows. For all $C, D \in \mathcal{X}$,

$$C \succeq_{\triangle} D \Leftrightarrow |C \triangle G| \leq |D \triangle G| .$$

According to $\succeq_{\triangle}$, if C and D contain the same number of good objects, C will be at least as good as D if and only if the number of bad objects in C is less than the corresponding number for D. Analogously, if C and D contain the same number of bad objects, C will be at least as good as D if and only if the number of good objects in C is weakly greater than the corresponding number for D. Finally, if C and D have different numbers of good and bad objects, the rule adds the number of bad objects in C (D) to the number of good objects which are not in C (D) in order to produce a comparison.

We introduce now the following three axioms a binary relation $\succeq \in \mathcal{P}$ may satisfy:

Perfect Dichotomy (PD) iff, for all $C, D \in \mathcal{X}$, all $x \in C \cap G$, and all $y \in B \setminus D$,
$C \succeq D \Leftrightarrow C \setminus \{x\} \succeq D \cup \{y\}$;

Good Influence (GINF) iff, for all $C, D \in \mathcal{X}$, all $x \in G \setminus C$,
$C \sim D \Rightarrow C \cup \{x\} \succ D$.

Bad Influence (BINF) iff, for all $C, D \in \mathcal{X}$, all $x \in B \setminus D$,
$C \sim D \Rightarrow C \succ D \cup \{x\}$.

Perfect dichotomy states that "being good" and "not being bad" have the same meaning for the decision maker, i.e. the "marginal rate of substitution" between good and bad objects is -1. According to this axiom, if a set of objects C is at least as good as another set of objects D, then substracting a good object from C and adding a bad object to D do not change the comparison of the sets.

Good influence requires that if two sets of objects are indifferent, adding a good object to one of them makes that set strictly better. This axiom was introduced by Barberà et al. (1991) and, more explicitly, by Fishburn (1992). The interpretation of our last axiom, bad influence, is analogous.

3 Characterization result

In proving our characterization result we shall make use of the following three lemmas.

Lemma 1 *If $\succeq \in \mathcal{P}$ satisfies GINF and PD, then it also satisfies BINF.*

Proof. We have to prove that for all $C, D \in \mathcal{X}$ and for all $z \in B \setminus D$, $C \sim D$ implies $C \succ D \cup \{z\}$.

Take C, D and z as above. We distinguish the following two possibilities:

(1) $C \cap G \neq \emptyset$. Let $x \in C \cap G$. Then $C \sim D \Rightarrow_{PD} C \setminus \{x\} \sim D \cup \{z\} \Rightarrow_{GINF} C \succ D \cup \{z\}$.

(2) $C \subseteq B$. Let $x \in G$. Then $C \sim D \Rightarrow_{GINF} C \cup \{x\} \succ D \Rightarrow_{PD} C \succ D \cup \{z\}$.

From these two cases we conclude that $\succeq$ satisfies BINF. ■

Remark 1 *Similarly, one can show that BINF and PD imply GINF.*

As mentioned in the Introduction, there are contexts in which the specification of good and bad objects implies homogeneity and full substitutability of the objects within a particular group. Our next lemma relates those contexts to perfect dichotomy.

Lemma 2 *If $\succeq \in \mathcal{P}$ satisfies PD, then, for all $C \in \mathcal{X}$,*
(1) $C \sim (C \setminus \{x\}) \cup \{y\}$ *for all* $x \in C \cap G$ *and* $y \in G \setminus C$;
(2) $C \sim (C \setminus \{x\}) \cup \{y\}$ *for all* $x \in C \cap B$ *and* $y \in B \setminus C$.

Proof. We show only (1). The proof of (2) is similar.

Suppose $C \in \mathcal{X}$ is such that $C \cap G \neq \emptyset$ and $G \setminus C \neq \emptyset$. Let $x \in C \cap G$ and $y \in G \setminus C$, and let $z \in B$. Notice that such a z exists by assumption. We distinguish the following two cases:

(1.a) $z \in C$. Then $C \succeq C \Leftrightarrow C \cup \{y\} \succeq C \setminus \{z\} \Leftrightarrow (C \cup \{y\}) \setminus \{x\} \succeq C$. Moreover, $(C \cup \{y\}) \setminus \{x\} \succeq (C \cup \{y\}) \setminus \{x\} \Leftrightarrow C \cup \{y\} \succeq (C \cup \{y\}) \setminus \{x, z\} \Leftrightarrow C \succeq (C \cup \{y\}) \setminus \{x\}$. Hence, $C \sim (C \setminus \{x\}) \cup \{y\}$.

(1.b) $z \notin C$. Then $C \succeq C \Leftrightarrow C \setminus \{x\} \succeq C \cup \{z\} \Leftrightarrow (C \setminus \{x\}) \cup \{y\} \succeq C$ and $(C \setminus \{x\}) \cup \{y\} \succeq (C \setminus \{x\}) \cup \{y\} \Leftrightarrow C \setminus \{x\} \succeq (C \setminus \{x\}) \cup \{y, z\} \Leftrightarrow C \succeq (C \setminus \{x\}) \cup \{y\}$. Therefore, $C \sim (C \setminus \{x\}) \cup \{y\}$. ■

Remark 2 *Notice that, by transitivity of $\succeq$ and the repeated use of Lemma 2, PD guarantees that the decision maker will be indifferent between any two sets consisting either of the same number of (possibly different) good objects or of the same number of (possibly different) bad objects.*

Lemma 3 *Suppose $\succeq \in \mathcal{P}$ satisfies PD. If $C, D \in \mathcal{X}$ are such that $C \subseteq D$ and $D \setminus C = \{x, y\}$ with $x \in G$ and $y \in B$, then $C \sim D$.*

Proof. Take $C, D \in \mathcal{X}$ as above. Then $C \cup \{x\} = D \setminus \{y\}$ and $C \cup \{y\} = D \setminus \{x\}$. Hence, $C \cup \{x\} \sim D \setminus \{y\}$ and $C \cup \{y\} \sim D \setminus \{x\}$ by reflexivity of $\succeq$. From PD, we have $C \succeq D$ and $D \succeq C$ and, therefore, $C \sim D$. ■

We are now ready for our characterization result.

Theorem 1 *A binary relation $\succeq \in \mathcal{P}$ satisfies PD and GINF if and only if $\succeq = \succeq_\triangle$.*

Proof. It is straightforward to verify that $\succeq_\triangle$ satisfies both axioms. In what follows we show that if a binary relation $\succeq \in \mathcal{P}$ satisfies PD and GINF, then $\succeq$ is uniquely determined.

For each $m \in \{0, 1, \ldots, |G|\}$ and for each $n \in \{1, \ldots, |B|\}$, define the families $\mathcal{G}_m$ and $\mathcal{B}_n$ as follows:

$$\mathcal{G}_m = \{G_m \subseteq G : |G_m| = m\}, \mathcal{B}_n = \{B_n \subseteq B : |B_n| = n\}.$$

Notice that, in view of Remark 2, the decision maker is indifferent between any two sets in the family $\mathcal{G}_m$ for each $m \in \{0, 1, \ldots, |G|\}$, and that he is also indifferent between any two sets in the family $\mathcal{B}_n$ for each $n \in \{1, \ldots, |B|\}$.

Moreover, by transitivity of $\succeq$ and by the repeated use of GINF and BINF (via Lemma 1), we have the following ranking for each $G_m \in \mathcal{G}_m$ and each $B_n \in \mathcal{B}_n$:

$$G = G_{|G|} \succ \ldots \succ G_1 \succ G_0 = \emptyset \succ B_1 \succ \ldots \succ B_{|B|} = B.$$

Consider now a set $C \in \mathcal{X}$ with $|C \cap G| = k \geq 1$ and $|C \cap B| = l \geq 1$. Without loss of generality, let $C \cap G = \{c_1^g, \ldots, c_k^g\}$ and $C \cap B = \left\{c_1^b, \ldots, c_l^b\right\}$. If $k \geq l$, then, by transitivity of $\succeq$ and by the repeated use of Lemma 3, we have $C \sim C \setminus \left\{c_1^g, \ldots, c_l^g, c_1^b, \ldots, c_l^b\right\} \in \mathcal{G}_{k-l}$. If $l > k$, then, by the same argument, we have $C \sim C \setminus \left\{c_1^g, \ldots, c_k^g, c_1^b, \ldots, c_k^b\right\} \in \mathcal{B}_{l-k}$. ■

Remark 3 *In view of Remark 1, it follows from the proof of Theorem 1 that* $\succeq_\triangle$ *is characterized by PD and BINF as well.*

References

[1] Barberà, S., W. Bossert, and P. K. Pattanaik (2003): Ranking sets of objects, in: Barberà, S., P. J. Hammond, and Ch. Seidl (eds.), Handbook of Utility Theory, Volume 2, Kluwer Academic Publishers, forthcoming.

[2] Barberà, S., H. Sonnenschein, and L. Zhou (1991): Voting by committees, Econometrica 59, 595-609.

[3] Berga, D., G. Bergantiños, J. Massó, and A. Neme (2003): Stability and voting by committees with exit, Social Choice and Welfare, forthcoming.

[4] Burani, N. and W. Zwicker (2003): Coalition formation games with separable preferences, Mathematical Social Sciences 45, 27-52.

[5] Fishburn, P. C. (1992): Signed orders and power set extensions, Journal of Economic Theory 56, 1-19.

[6] Ju, B.-G. (2003): A characterization of strategy-proof voting rules for separable weak orderings, Social Choice and Welfare 21(3), 469-499.

[7] Kasher, A. and A. Rubinstein (1997): On the question "Who is a J?": a social choice approach, Logique et Analyse 160, 385-395.

[8] Pattanaik, P. and Y. Xu (1990): On ranking opportunity sets in terms of freedom of choice, Recherches Economiques de Louvain 56, 383-390.

[9] Samet, D. and D. Schmeidler (2003): Between liberalism and democracy, Journal of Economic Theory 110, 213-233.