E C O N O M I C S   B U L L E T I N

Path independence and Pareto dominance

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Abstract

A choice function on a finite set is path independent if and only if it can be represented as the choice of undominated alternatives taking into account mixed strategy domination.

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1. Introduction

The notion of a path independent choice function was introduced by Plott (1973). The property is often considered a minimal requirement for rational choice; it is satisfied, in particular, if maximal elements of a given strict order are chosen from every subset (however, the choice of maximizers for an acyclic relation need not be path independent). Aizerman and Malishevski (1981) observed that every path independent choice function admits a joint-extremal representation, i.e., maximal elements of a given family of linear orders are chosen from every subset. Apparently the first published proof is dispersed in Aleskerov et al. (1979) and Litvakov (1980, 1981); Moulin (1985) provides a closed proof.

This paper gives another characterization of path independent functions: they can be represented by the choice of undominated alternatives taking into account mixed strategy domination. In a sense, an elementary explanation of the connection between path independence and convex structures (Koshevoy, 1999; Monjardet and Raderanirina, 2001; Danilov and Koshevoy, 2003) is provided although this simple theorem cannot claim to have exhausted the topic.

2. The Result

Let A be a finite set, $\#A = n$, and $\mathfrak{A} = 2^A \setminus \{\emptyset\}$. A choice function on A is a mapping $C : \mathfrak{A} \to \mathfrak{A}$ satisfying $C(X) \subseteq X$ for every $X \in \mathfrak{A}$. A choice function is path independent if $C(X \cup Y) = C(C(X) \cup C(Y))$ for all X and Y.

Let there be a finite list of functions $P_i : A \to \mathbb{R}$, $i \in N$; we say that a choice function C on A is Pareto represented (with vector criterion $\langle P_i \rangle_{i \in N}$) if

$$\forall X \in \mathfrak{A}\, \forall x \in X \Big[x \in C(X) \iff \nexists\, y^1, \ldots, y^m \in X,\ \lambda_1, \ldots, \lambda_m \in \mathbb{R} \\ \Big[\sum_k \lambda_k = 1 \;\&\; \forall k\, [\lambda_k \geq 0] \;\&\; \forall i \in N \Big(\sum_k \lambda_k P_i(y^k) > P_i(x) \Big) \Big] \Big].$$

Theorem. *A choice function on a finite set is path independent if and only if it can be Pareto represented (with a finite list of functions).*

Proof. The sufficiency is well known (and easy to check anyway); we only consider the necessity.

We denote Σ the set of one-to-one mappings $\sigma : \{1, \ldots, n\} \to A$. For every $\sigma \in \Sigma$, we define $\sigma_* \in \Sigma$ as follows: $\sigma_*(1) = \operatorname{argmin}_{x \in C(A)} \sigma^{-1}(x)$, $A(1) = A$; for $k = 1, \ldots, n-1$, by induction, $A(k+1) = A(k) \setminus \{\sigma_*(k)\}$ and $\sigma_*(k+1) = \operatorname{argmin}_{x \in C(A(k+1))} \sigma^{-1}(x)$. Then we define $P_\sigma : A \to \mathbb{R}$ by

$$P_\sigma(x) = n^{[n - \sigma_*^{-1}(x)]}. \tag{1}$$

Let $x \in X \setminus C(X)$ for $X \in \mathfrak{A}$ and $C(X) = \{y^1, \ldots, y^m\}$; we define $\lambda_k = 1/m$ for $k = 1, \ldots, m$. Let $\sigma \in \Sigma$; if $\sigma_*^{-1}(y^h) < \sigma_*^{-1}(x)$ for some h, then $\frac{1}{m}\sum_k P_\sigma(y^k) > \frac{1}{n}P_\sigma(y^h) \geq P_\sigma(x)$. Suppose that $\sigma_*^{-1}(x) < \sigma_*^{-1}(y^k)$ for all $k = 1, \ldots, m$; then $x \in C(A(\sigma_*^{-1}(x)))$ while $y^k \in A(\sigma_*^{-1}(x))$ for all k. On the other hand, $C(A(\sigma_*^{-1}(x))) = C(C(\{y^1, \ldots, y^m, x\}) \cup C(A(\sigma_*^{-1}(x)) \setminus \{y^1, \ldots, y^m, x\}))$; x cannot belong to the second term, but it also does not belong to the first one: otherwise, we would have $C(X) = C(X \cup \{y^1, \ldots, y^m, x\}) = C(\{y^1, \ldots, y^m, x\}) \ni x$. The contradiction proves $\frac{1}{m}\sum_k P_\sigma(y^k) > P_\sigma(x)$ for all $\sigma \in \Sigma$.

Finally, let $\sum_{k=1}^m \lambda_k P_\sigma(y^k) > P_\sigma(x)$ for all $\sigma \in \Sigma$ and $x \neq y^k$ for any k. We have to show that $x \notin C(X)$ whenever $\{y^1, \ldots, y^m\} \subseteq X$. Let $\sigma \in \Sigma$ be such that $\sigma^{-1}(y^k) > \sigma^{-1}(z)$ for any k and any $z \notin \{y^1, \ldots, y^m\}$, and let $s = \min_k \sigma_*^{-1}(y^k)$. The supposed domination implies $P_\sigma(y^k) > P_\sigma(x)$ for some k, hence $s < \sigma_*^{-1}(x)$. Therefore, $A(s) \supseteq \{y^1, \ldots, y^m, x\}$ and $C(A(s)) = Y \subseteq \{y^1, \ldots, y^m\}$ because otherwise a $z \in Y \setminus \{y^1, \ldots, y^m\}$ would have been chosen as $\sigma_*(s)$. If it were that $x \in C(Y \cup \{x\})$, we would have $C(A(s)) = C(A(s) \cup [Y \cup \{x\}]) = C(Y \cup C(Y \cup \{x\})) = C(Y \cup \{x\}) \ni x$. Now $X = (Y \cup \{x\}) \cup [X \setminus (Y \cup \{x\})]$ implies $x \notin C(X)$. □

3. Concluding Remarks

1. All the functions $P_\sigma(\cdot)$ are one-to-one; therefore, weak Pareto dominance

$$\forall \sigma \in \Sigma \left[\sum_k \lambda_k P_\sigma(y) \geq P_\sigma(x)\right] \;\&\; \exists \sigma \in \Sigma \left[\sum_k \lambda_k P_\sigma(y) > P_\sigma(x)\right]$$

produces the same choice function C. It is also worthwhile to note that $P = \langle P_\sigma \rangle_{\sigma \in \Sigma}$ embeds A into $\mathbb{R}^\Sigma$.

2. It is easy to check that the same functions $P_\sigma(\cdot)$ provide a joint-extremal representation of C. Moreover, (1) shows how any joint-extremal representation can be converted into a Pareto representation.

3. When perceived as a closed proof of the joint-extremal representation, the above reasoning may be the shortest in the literature. This advantage is paid for with our total disregard for the number of dimensions involved: for instance, if $\sigma'_* = \sigma''_*$, then the functions $P_{\sigma'}(\cdot)$ and $P_{\sigma''}(\cdot)$ are identical; if $C(A(s)) = A(s)$ for some s, there is no need to define $\sigma_*(s+1), \ldots, \sigma_*(n)$; etc. Consistent economizing would lead to a proof indistinguishable from Moulin's (1985).

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