E C O N O M I C S B U L L E T I N

Orthogonal Subgroups for Portfolio Choice

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Abstract

The orthogonal group on the location−scale family is at the foundation of the stochastic structure underlying CAPM. Relaxing that assumption, we show how less restrictive matrix subgroup symmetries on the location−scale family of asset returns bound asset choices. Sign symmetry is a special case and provides conditions such that the investor does not sell short. Group−generated welfare orderings are also identified

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1. Introduction

The inherent symmetries of the investor's portfolio allocation problem have long been recognized. These symmetries are exploited through the normal and elliptical symmetry assumptions and are central to understanding when the capital asset pricing model applies. Chamberlain (1983), Owen and Rabinovitch (1983), Madan and Seneta (1990), and Berk (1997) explain details on how elliptical symmetry and the capital asset pricing model relate. The symmetries attending the normal and elliptically symmetric distributions are infinite because the invariances may be represented by an infinitely large set of matrix operations. Lapan and Hennessy (2002) and Hennessy and Lapan (2003) have studied portfolio allocation under less complete symmetries. These symmetries, the symmetry group on a finite number of objects and arbitrary subgroups of that group, may be viewed as subgroups of a larger group, the orthogonal group, that generates elliptically symmetric distributions.

This note demonstrates how to work with arbitrary subgroups of the orthogonal group when studying the effects of symmetries on the structure of portfolio allocations. To illustrate, we provide choice bounds for the group of sign symmetries and we show what this group means for the decision of whether to short an asset. We will demonstrate that the bounds we identify have strong welfare implications. If Sharpe indices become more dispersed in a matrix group determined sense, then an investor's ex-ante welfare increases.

2. Orthogonal group and elliptical symmetry

The orthogonal group defines the family of elliptically symmetric distributions.

DEFINITION 1. *The orthogonal group of dimension N is i) the set of all $N \times N$ matrices with real entries such that any element M satisfies $M^t \times M = I_N$, with superscripted t as the transpose operation and I_N as the identity in $N \times N$ matrices, together with ii) the matrix multiplication operation. The orthogonal matrix set is labeled $\mathcal{O}_N$ and the group under matrix multiplication is labeled $\tilde{\mathcal{O}}_N$.*[1]

The group is uncountably infinite in the sense that $\mathcal{O}_N$ is an uncountably infinite set. For example there are uncountably infinite distinct rotations around the origin in $\mathbb{R}^2$, and all can be represented by elements in $\mathcal{O}_N$.

DEFINITION 2. *$N \times 1$ random vector ε follows a spherical distribution if for all $M \in \mathcal{O}_N$ the invariance $M\varepsilon =^d \varepsilon$ applies where $=^d$ means equal in distribution.*

DEFINITION 3. *$N \times 1$ random vector x follows the elliptically symmetric distribution with $N \times 1$ location parameter vector μ and $N \times N$ dispersion parameter matrix Ψ if $x =^d \mu + C^t\varepsilon$ where ε follows a spherical distribution, C is a $k \times N$ matrix, $C^tC = \Psi$, and Ψ has rank k.*

DEFINITION 4. *A matrix subgroup of the orthogonal group under matrix multiplication is a (topologically) closed subset G of $\mathcal{O}_N$ that is closed under matrix multiplication. The matrix subgroup is written as $\tilde{G}$ and the subgroup relation is written as $\tilde{G} \sqsubseteq \tilde{\mathcal{O}}_N$.*

DEFINITION 5. *A permutation subgroup of $\tilde{\mathcal{O}}_N$ is a group $\tilde{G} \sqsubseteq \tilde{\mathcal{O}}_N$ such that each matrix element in G contains one and only one number 1 in each row and in each column, while the*

[1] Materials related to definitions 1 through 3 may be viewed in Fang, Kotz, and Ng (1990). See Armstrong (1988) on definitions 4 and 6.

remaining entries have value 0.

DEFINITION 6. *For $\tilde{G} \sqsubseteq \tilde{\mathcal{O}}_N$, distribution $F(\varepsilon)$ is said to be $\tilde{G}$-symmetric if $M\varepsilon =^d \varepsilon$ $\forall$ $M \in G$.*

3. Model

An expected utility maximizer at time point 0 with time point 1 payoff W and monotone increasing time 1 utility function $U(W): \mathbb{R} \to \mathbb{R}$ has available \$1 to invest in any amount of $N+1$ assets. One of these assets is risk-free with return r. This is the rate the agent can borrow and lend at. The other N assets are risky with location-and-scale parameterized rates of return;

$$x_n = \mu_n + \sigma_n \varepsilon_n, \quad \sigma_n > 0 \ \forall n \in \{1, 2, \ldots, N\} \equiv \Omega_N. \tag{1}$$

The distribution of vector ε is given as $F(\varepsilon)$ with expectation operator $E[\cdot]$ and $E[\varepsilon_n] = 0$ $\forall n \in \Omega_N$.[2] The agent allocates $\$a_n$ to the nth risky asset, where $a \in \mathbb{R}^N$ is the vector of these allocations. Residual fund amount, positive or negative, is $1 - a^t \vec{1}$ where $\vec{1} \in \mathbb{R}^N$ is the vector of ones. This residual amount is allocated to the risk-free asset so that the budget is exhausted.

The agent's problem then is to

$$\max_a E\left\{U\left[r + a^t(\mu - r\vec{1}) + b^t\varepsilon\right]\right\}, \qquad b = (a_1\sigma_1, a_2\sigma_2, \ldots, a_N\sigma_N)^t, \tag{2}$$

and denote maximizing arguments by solution $\hat{a}$. Throughout the analysis is written as if there is a unique optimizer. The analysis carries through for non-singleton optimizer sets. For θ as the Sharpe index vector with entries $\theta_n = (\mu_n - r)/\sigma_n$, the problem may be re-written as

$$\max_b E\left\{U\left[r + b^t\theta + b^t\varepsilon\right]\right\} \tag{3}$$

and solution vector $\hat{b}$ must satisfy $\hat{b}_n = \hat{a}_n\sigma_n$. The analysis that follows extends the permutation group analysis in Lapan and Hennessy (2002) and Hennessy and Lapan (2003) to the larger set of arbitrary orthogonal matrix groups.

If $F(\varepsilon)$ is $\tilde{G}$-symmetric, then

$$E\left\{U\left[r + \hat{b}^t\theta + \hat{b}^t\varepsilon\right]\right\} = E\left\{U\left[r + \hat{b}^t\theta + \hat{b}^t M\varepsilon\right]\right\} = E\left\{U\left[r + \hat{b}^t\theta + \tilde{b}^t\varepsilon\right]\right\}, \qquad \tilde{b}^t = (M^t\hat{b})^t. \tag{4}$$

But optimality and symmetry require that

$$\begin{aligned} &E\left\{U\left[r + \hat{b}^t\theta + \hat{b}^t\varepsilon\right]\right\} \ge E\left\{U\left[r + \tilde{b}^t\theta + \tilde{b}^t\varepsilon\right]\right\} = E\left\{U\left[r + \tilde{b}^t\theta + (M^t\hat{b})^t\varepsilon\right]\right\} \\ &= E\left\{U\left[r + \tilde{b}^t\theta + \hat{b}^t M\varepsilon\right]\right\} = E\left\{U\left[r + \tilde{b}^t\theta + \hat{b}^t\varepsilon\right]\right\} = E\left\{U\left[r + \hat{b}^t\theta + (\tilde{b}^t - \hat{b}^t)\theta + \hat{b}^t\varepsilon\right]\right\}. \end{aligned} \tag{5}$$

Comparing (4) and (5), the only difference in the payoff function is $(\tilde{b}^t - \hat{b}^t)\theta$. Optimality and invariance due to $\tilde{G}$-symmetry ensure that the payoff's stochastic attributes, as captured in $\hat{b}^t\varepsilon$, are exactly the same apart from the payoff location shift. The monotonicity property on $U(W)$

[2] Comparing (1) with definition 3, our model does not allow for full elliptical symmetry even when the symmetries of vector ε are $\tilde{\mathcal{O}}_N$. This is because (1) requires C to be diagonal. The model can be extended to full elliptical symmetry for full-rank square matrices C through a diagonalizing re-specification of the assets so that the note's findings would carry through up to a basis re-specification.

requires that $\tilde{b}^t\theta \le \hat{b}^t\theta$ or $(M^t\hat{b})^t\theta \equiv \hat{b}^t M\theta \le \hat{b}^t\theta \; \forall M \in G$. More formally,

PROPOSITION 1. *If $U(W): \mathbb{R} \to \mathbb{R}$ is increasing and $F(\varepsilon)$ is $\tilde{G}$-symmetric with $\tilde{G} \sqsubseteq \tilde{\mathcal{O}}_N$ then $\hat{b}^t(M - I_N)\theta \le 0 \; \forall M \in G$.*

Each $M \in G$ creates a bound that $\hat{b}$ must satisfy. Lapan and Hennessy (2002) and Hennessy and Lapan (2003) provide permutation group illustrations of these bounds, but permutations do not allow for axial reflections, arbitrary rotations around the origin or reflections through arbitrary rays from the origin.

3.1. Elliptical Symmetry. If $\tilde{G} = \tilde{\mathcal{O}}_N$ in proposition 1, then the group supports elliptical symmetry. This example is of particular interest because the set $M^t\hat{b} \; \forall M \in \mathcal{O}_N$ orbits (i.e., re-locates) $\hat{b}$ over the entire sphere centered at the origin in $\mathbb{R}^N$ and containing the optimum, $\hat{b}$;

$$\sum\nolimits_{n=1}^{N} b_n^2 = \kappa, \qquad \kappa = \sum\nolimits_{n=1}^{N} \hat{b}_n^2. \qquad (6)$$

Equation (6) demonstrates the dimension of the orthogonal group; its matrices can map one point on a non-degenerate sphere to any other point on it. The level of symmetry that elliptical symmetry imposes gauges the assumption's restrictiveness. Equation (6) also demonstrates the utility of linear, i.e., matrix, groups in portfolio problems. The linearity of $b^t\varepsilon$ means that a linear invariance on $F(\varepsilon)$ implies some invariance consequence on b through the relation $b^t\varepsilon = b^t M M^{-1}\varepsilon$.

In light of (6), another way of writing condition set $(M^t\hat{b})^t\theta \le \hat{b}^t\theta \; \forall M \in \mathcal{O}_N$ is as the constrained optimization problem

$$\hat{b} = \arg\max_b b^t\theta + \lambda \times \left(\kappa - \sum\nolimits_{n=1}^{N} b_n^2\right), \qquad \kappa = \sum\nolimits_{n=1}^{N} \hat{b}_n^2; \qquad (7)$$

where λ is a Lagrange multiplier. The investor could choose any point on the sphere and chooses $\hat{b}$ as the sphere-constrained optimum. The presence of a risk-free asset ensures that there is no budget constraint on b, and so the sphere can have arbitrarily large radius. Optimization on (7) provides $\theta_i / \theta_j = \hat{b}_i / \hat{b}_j$ so that allocations among risky assets are in direct proportion to the Sharpe indices and the only allocation vectors one need consider are along the ray $b = \bar{\theta} \times (\theta_1, \theta_2, \ldots, \theta_N)^t, \bar{\theta} \in \mathbb{R}$, as a subspace in $b \in \mathbb{R}^N$. The way that groups enter this optimization problem is a particularly simple application of Lie (continuous) groups where the orthogonal group generates the sphere manifold. Baker (2002) provides an accessible introduction to matrix representations of Lie groups.

3.2. Sign Symmetry. Interesting subgroups of $\tilde{\mathcal{O}}_N$ are the groups of sign changes, i.e., reflection symmetries through axes. For density $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$, the group of all sign changes is defined by the invariances $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \equiv f(-\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \equiv f(\varepsilon_1, -\varepsilon_2, \ldots, \varepsilon_N) \equiv \ldots \equiv f(\varepsilon_1, \varepsilon_2, \ldots, -\varepsilon_N)$ together with the invariances these sign change operations generate. The relevant matrices are of form

$$\begin{pmatrix} (-1)^{\delta_1} & 0 & \vdots & 0 \\ 0 & (-1)^{\delta_2} & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & (-1)^{\delta_N} \end{pmatrix}, \qquad \delta_n \in \{0,1\}, \qquad n \in \Omega_N; \tag{8}$$

and there are 2^N such matrices, which we label as set $\mathfrak{D}_N$ with associated matrix group $\tilde{\mathfrak{D}}_N$.

Applying proposition 1, if $\tilde{G} = \tilde{\mathfrak{D}}_N$ then $\hat{b}_n \times (\mu_n - r)/\sigma_n \geq 0 \; \forall n \in \Omega_N$ and $\hat{b}_n \geq 0$ whenever $\mu_n \geq r$. If some $\mu_n \geq r$ then it will not be optimal for any investor with non-satiated preferences to short that *n*th asset. Notice that statistical independence has been assumed nowhere in the analysis. But $\text{cov}(\varepsilon_i, \varepsilon_j) = 0 \; \forall i, j \in \Omega_N$ under $\tilde{G} = \tilde{\mathfrak{D}}_N$ because $E[\varepsilon_i \varepsilon_j] = E[-\varepsilon_i \varepsilon_j]$ implies $E[\varepsilon_i \varepsilon_j] = 0$. On the other hand, if $\tilde{G} \sqsubseteq \tilde{\mathfrak{D}}_N$ with G comprised of just I_N and $-I_N$ then proposition 1 only asserts that $\sum_{n=1}^{N} \hat{b}_n \times (\mu_n - r)/\sigma_n \geq 0$. In that case, if $\mu_n \geq r \; \forall n \in \Omega_N$ then at least one $\hat{b}_n$ must be non-negative.

4. Welfare

Proposition 1 provides insights into understanding how asset means and variances in our model affect ex-ante welfare. To see this, define the convex hull of the orbit of vector $z' \in \mathbb{R}^N$ under matrix group $\tilde{G} \sqsubseteq \tilde{\mathcal{O}}_N$ as $\text{H}(z'; \tilde{G})$. If $z'' \in \text{H}(z'; \tilde{G})$ then z'' is said to be $\tilde{G}$-majorized by z', and we write $z'' \leq_{\tilde{G}} z'$. See p. 422 of Marshall and Olkin (1979) or Mudholkar (1966) for details on group majorization.

The most well-known case of $\tilde{G}$-majorization is ordinary majorization, which is the underpinning of Lorenz curve dominance and other measures of income dispersion. In that case, the group is the group of all permutations on a set of objects. Our interest is in the role of Sharpe index dispersion on the investor's ex-ante welfare. Proposition 4.1 in Lapan and Hennessy (2002) shows that more Sharpe index dispersion, in the sense of ordinary majorization, increases ex-ante welfare for a risk averse investor. In this section we extend their result to arbitrary matrix subgroups of the orthogonal group.

The $\tilde{G}$-majorization pre-ordering is of interest because of the $\tilde{G}$-increasing property. From Eaton and Perlman (1977, p. 830), a function $f(z)$ is said to be $\tilde{G}$-increasing if $z'' \leq_{\tilde{G}} z'$ implies $f(z'') \leq f(z')$. To establish why the property is of relevance, assume a strictly concave utility function and define maximized ex-ante welfare as

$$S(\theta) = \max_b E\left\{U\left[r + b^t\theta + b^t\varepsilon\right]\right\}. \tag{9}$$

This function is symmetric in the sense that if $F(\varepsilon)$ is $\tilde{G}$-symmetric then $S(\theta)$ is also $\tilde{G}$-symmetric, $S(\theta) \equiv S(M\theta) \; \forall M \in G$. The envelope theorem provides derivative $\partial S(\theta)/\partial \theta_n = \hat{b}_n E\{U_W(W)\}$, where $U_W(W)$ is marginal utility. Write the entries in vector $M\theta$ as $\theta_1(M)$ through $\theta_N(M)$.

With strictly increasing utility function, proposition 1 implies

$$\sum_{n=1}^{N} \theta_n(M) \frac{\partial S(\theta)}{\partial \theta_n} \le \sum_{n=1}^{N} \theta_n(I_N) \frac{\partial S(\theta)}{\partial \theta_n} \quad \forall M \in G. \tag{10}$$

Eaton and Perlman (1977, p. 830) show that if $S(\theta)$ is $\tilde{G}$-symmetric and (10) is true on the domain of θ then function $S(\theta)$ is $\tilde{G}$-increasing. Thus we know that the investor's ex-ante welfare as a function of Sharpe indices, $S(\theta)$, has the property of $\tilde{G}$-increasing, $\tilde{G} \sqsubseteq \tilde{\mathcal{O}}_N$.

PROPOSITION 2. *If $U(W): \mathbb{R} \to \mathbb{R}$ is increasing, continuously differentiable, and strictly concave while $F(\varepsilon)$ is $\tilde{G}$-symmetric with $\tilde{G} \sqsubseteq \tilde{\mathcal{O}}_N$ then $S(\theta'') \le S(\theta')$ whenever $\theta'' \le_{\tilde{G}} \theta'$.*

We can go further upon making the assumption that all Sharpe indices are non-negative, but the notion of association is required. The vector of random variables ε is said to be associated if $\mathrm{cov}(f_1(x), f_2(x)) \ge 0$ for all pairs of increasing functions $f_1(x)$ and $f_2(x)$. The property is of relevance because the first-order condition in the portfolio problem may be written as

$$\theta_n = -\frac{E\left\{U_W[r + b^t\theta + b^t\varepsilon]\varepsilon_n\right\}}{E\left\{U_W[r + b^t\theta + b^t\varepsilon]\right\}} \quad \forall n \in \Omega_N. \tag{11}$$

If ε is associated, the utility function is strictly concave, and all Sharpe indices are non-negative, then $\hat{b}_n \ge 0 \; \forall n \in \Omega_N$ and $\partial S(\theta)/\partial \theta_n \ge 0 \; \forall n \in \Omega_N$. Define $z'' \le_{\tilde{G}}^{w} z'$ whenever there exists a $z''' \in \mathbb{R}^N$ such that $z'' \le_{\tilde{G}} z'''$ and $z_n''' \le z_n' \; \forall n \in \Omega_N$. If $\theta'' \le_{\tilde{G}}^{w} \theta'$, random variables ε are associated, the investor is strictly risk averse and all Sharpe indices are positive, then there exists a θ''' such that $S(\theta''') \le S(\theta')$ (by monotonicity) and also $S(\theta'') \le S(\theta''')$ (by proposition 2).

PROPOSITION 3. *Let $U(W): \mathbb{R} \to \mathbb{R}$ be increasing, continuously differentiable, and strictly concave while $F(\varepsilon)$ is a $\tilde{G}$-symmetric distribution, $\tilde{G} \sqsubseteq \tilde{\mathcal{O}}_N$, for associated random variables ε. If all Sharpe indices are non-negative, then $S(\theta'') \le S(\theta')$ whenever $\theta'' \le_{\tilde{G}}^{w} \theta'$.*

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